Chapter – 10: Straight Line

Example 1

Find the slope of the lines:

(a) Passing through the points $(3,-2)$ and $(-1,4)$,

(b) Passing through the points $(3,-2)$ and $(7,-2)$,

(c) Passing through the points $(3,-2)$ and $(3,4)$,

(d) Making inclination of 60° with the positive direction of x-axis.

Solution

(a)

Decide the distinction in y-directions of these two focuses (rise).

Decide the distinction in x-facilitates for these two focuses (run).

Partition the distinction in y-facilitates by the distinction in x-organizes (rise/run or incline).

The slope of the line through $(3,-2)$ and $(-1,4)$ is

$$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{4-(-2)}{-1-3}=\frac{6}{-4}=-\frac{3}{2}$$

(b)

The formula for slope of a line is

$$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$$

Then

The slope of the line through the points $(3,-2)$ and $(7,-2)$ is

$$m=\frac{-2-(-2)}{7-3}=\frac{0}{4}=0$$

(c)

The general formula for the slope of a line is

$$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$$

Then

The slope of the line through the points $(3,-2)$ and $(3,4)$

$m=\frac{4-(-2)}{3-3}=\frac{6}{0}$, which is not defined.

(d) Here in this case

The inclination of the line $\alpha=60^{\circ}$.

Then, the slope of the line is $m=\tan 60^{\circ}=\sqrt{3}$

Example 2

If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution

Given that

The angle between two lines is $\frac{\pi}{4}$

The slope of one of the lines is $\frac{1}{2}$

We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right| \quad \ldots\ldots\ldots(1)$$

Let $m_1=\frac{1}{2}, m_2=m$

and $\theta=\frac{\pi}{4}$.

Substitute the values in the equation(1),

we get

$$\tan\frac{\pi}{4}=\left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right|$$

Simplifying

$$1=\left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right|$$

which gives

$$\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}=1 \text{ or } \frac{m-\frac{1}{2}}{1+\frac{1}{2}m}=-1$$

Therefore $m=3$ or $m=-\frac{1}{3}$.

Hence, slope of the other line is 3 or $-\frac{1}{3}$.

Example 3

Line through the points $(-2,6)$ and $(4,8)$ is perpendicular to the line through the points $(8,12)$ and $(x,24)$. Find the value of x.

Solution

The general expression for slope of a line is

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Slope of the line through the points $(-2,6)$ and $(4,8)$ is

$$m_1=\frac{8-6}{4-(-2)}=\frac{2}{6}=\frac{1}{3}$$

simillarly

Slope of the line through the points $(8,12)$ and $(x,24)$ is

$$m_2=\frac{24-12}{x-8}=\frac{12}{x-8}$$

If two lines are perpendicular,

The product of their slope is 1

Then

$$m_1m_2=-1,$$

which gives

$\frac{1}{3}\times\frac{12}{x-8}=-1$ or $x=4$

Example 4

Three points $P(h,k), Q(x_1,y_1)$ and $R(x_2,y_2)$ lie on a line.

Show that

$(h-x_1)(y_2-y_1)=(k-y_1)(x_2-x_1)$

Solution

If two points are collinear, the slope of the lines connecting them are equal.

Since points P, Q and R are collinear,

Then, we have

Slope of $PQ =$ Slope of QR,

i.e., $\frac{y_1-k}{x_1-h}=\frac{y_2-y_1}{x_2-x_1}$

or $\frac{k-y_1}{h-x_1}=\frac{y_2-y_1}{x_2-x_1}$

Cross multiply the terms on each side.

Expanding the equation we have

$(h-x_1)(y_2-y_1)=(k-y_1)(x_2-x_1)$.

Hence proved

Example 5

In Fig 10.9, time and distance graph of a linear motion is given. Two positions of time and distance are recorded as, when $T=0, D=2$ and when $T=3, D=8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

Solution

First we have to consider a point on the line,

Let (T, D) be any point on the line,

where D denotes the distance at time T.

Therefore,

The points $(0,2),(3,8)$ and (T, D) are collinear

If two points are collinear, the slope of the lines connecting them are equal.

In general, three points A, B and C are collinear if the sum of the lengths of any two line segments among A B, B C and C A is equal to the length of the remaining line segment,

that is, either $AB+BC=AC$ or $AC+CB=AB$ or $BA+AC=BC$.

so that

$$\frac{8-2}{3-0}=\frac{D-8}{T-3}$$

Simplify

$$6(T-3)=3(D-8)$$

Divide both sides by 6

$$\frac{6(T-3)}{6}=\frac{3(D-8)}{6}$$

Simplify

$$T-3=\frac{D-8}{2}$$

Add 3 to both sides

$$T-3+3=\frac{D-8}{2}+3$$

Simplify

$$T=\frac{D-8}{2}+3$$

which is the required relation.

Exercise 10.1

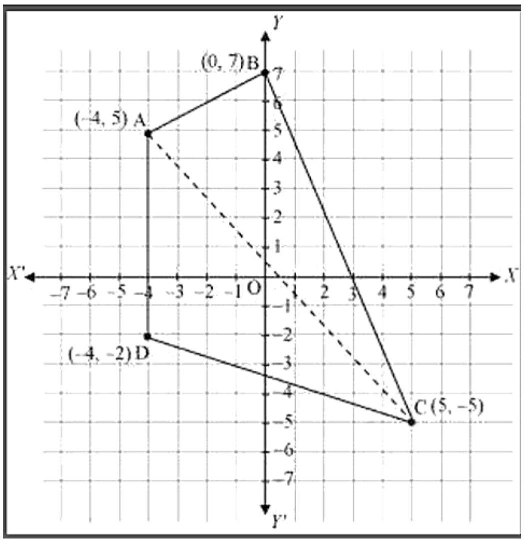
Question 1:

Draw a quadrilateral in the Cartesian plane, whose vertices are $(-4,5),(0,7),(5,-5)$ and $(-4,-2)$. Also, find its area.

Answer 1:

Let A B C D be the given quadrilateral with vertices $A(-4,5), B(0,7), C(5,-5)$, and $D(-4,-2)$.

Then, by plotting A, B, C, and D on the Cartesian plane and joining A B, B C, C D, and D A, the given quadrilateral can be drawn as



To find the area of quadrilateral A B C D, we draw one diagonal, say A C.

Accordingly, area $(ABCD) =$ area $(\triangle ABC) +$ area $(\triangle ACD)$

We know that the area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) is

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Therefore, area of $\triangle$ABC

$$= \frac{1}{2}|-4(7+5) + 0(-5-5) + 5(5-7)| \text{ unit}^2$$

$$= \frac{1}{2}|-4(12) + 5(-2)| \text{ unit}^2$$

$$= \frac{1}{2}|-48 - 10| \text{ unit}^2$$

$$= \frac{1}{2}|-58| \text{ unit}^2$$

$$= \frac{1}{2} \times 58 \text{ unit}^2$$

$$= 29 \text{ unit}^2$$

Area of $\triangle ACD$

$$= \frac{1}{2}|-4(-5+2) + 5(-2-5) + (-4)(5+5)| \text{ unit}^2$$

$$= \frac{1}{2}|-4(-3) + 5(-7) - 4(10)| \text{ unit}^2$$

$= \frac{1}{2}|12-35-40|$ unit ^{2}

$= \frac{1}{2}|-63|$ unit ^{2}

$= \frac{63}{2}$ unit ^{2}

Thus, area $(ABCD) = \left(29+\frac{63}{2}\right)$ unit $^2 = \frac{58+63}{2}$ unit $^2 = \frac{121}{2}$ unit ^{2}

Question 2:

The base of an equilateral triangle with side $2 a$ lies along they y-axis such that the mid point of the base is at the origin. Find vertices of the triangle.

Answer 2:

Let A B C be the given equilateral triangle with side 2 a.

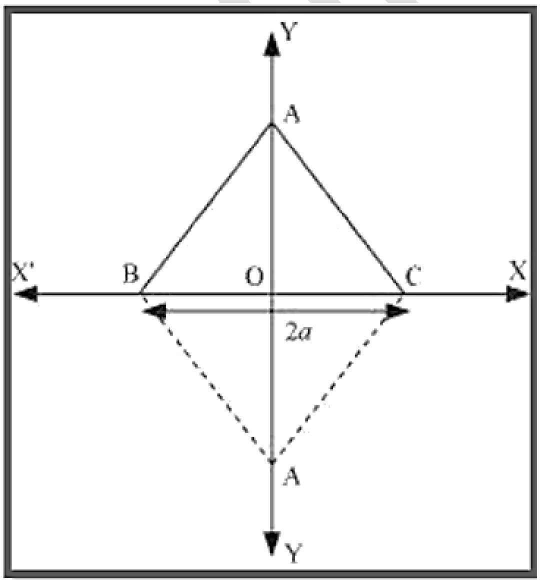
Accordingly, $AB = BC = CA = 2a$

Assume that base BC lies along the y-axis such that the mid-point of BC is at the origin. i.e., $\mathrm{BO} = \mathrm{OC} = a$, where O is the origin.

Now, it is clear that the coordinates of point C are $(0, a)$, while the coordinates of point B are $(0, -a)$

It is known that the line joining a vertex of an equilateral triangle with the mid-point of its opposite side is perpendicular.

Hence, vertex A lies on the y-axis.



On applying Pythagoras theorem to $\triangle \mathrm{AOC}$, we obtain

$$(\mathrm{AC})^2 = (\mathrm{OA})^2 + (\mathrm{OC})^2$$

$$\Rightarrow (2a)^2 = (OA)^2 + a^2$$

$$\Rightarrow 4a^2 - a^2 = (OA)^2$$

$$\Rightarrow (OA)^2 = 3a^2$$

$$\Rightarrow OA = \sqrt{3}a$$

$\therefore$ Coordinates of point $\mathrm{A} = (\pm\sqrt{3}a, 0)$

Thus, the vertices of the given equilateral triangle are $(0,a),(0,-a)$, and $(\sqrt{3}a,0)$ or $(0,a),(0,-a)$, and $(-\sqrt{3}a,0)$

Question 3:

Find the distance between $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$ when:

(i) PQ is parallel to the y-axis,

(ii) PQ is parallel to the x-axis.

Answer 3:

The given points are $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$.

(i) When PQ is parallel to the y-axis, $x_1 = x_2$.

In this case, distance between P and $\mathrm{Q} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(y_2 - y_1)^2}$$

$$= |y_2 - y_1|$$

(ii) When PQ is parallel to the x-axis, $y_1 = y_2$.

In this case, distance between P and $\mathrm{Q} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(x_2 - x_1)^2}$$

$$= |x_2 - x_1|$$

Question 4:

Find a point on the x-axis, which is equidistant from the points $(7,6)$ and $(3,4)$.

Answer 4:

Let $(a,0)$ be the point on the x axis that is equidistant from the points $(7,6)$ and $(3,4)$.

Accordingly, $\sqrt{(7-a)^2+(6-0)^2}=\sqrt{(3-a)^2+(4-0)^2}$

$\Rightarrow \sqrt{49+a^2-14a+36}=\sqrt{9+a^2-6a+16}$

$\Rightarrow \sqrt{a^2-14a+85}=\sqrt{a^2-6a+25}$

On squaring both sides, we obtain $a^2-14a+85=a^2-6a+25$

$\Rightarrow -14a+6a=25-85$

$\Rightarrow -8a=-60$

$\Rightarrow a=\frac{60}{8}=\frac{15}{2}$

Thus, the required point on the x-axis is $\left(\frac{15}{2},0\right)$.

Question 5:

Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $P(0,-4)$ and $B(8,0)$.

Answer 5:

The coordinates of the mid-point of the line segment joining the points

$P(0,-4)$ and $B(8,0)$ are $\left(\frac{0+8}{2},\frac{-4+0}{2}\right)=(4,-2)$

It is known that the slope (m) of a non-vertical line passing through the points (x_1,y_1) and (x_2,y_2) is given by $m=\frac{y_2-y_1}{x_2-x_1}, x_2\neq x_1$

Therefore, the slope of the line passing through $(0,0)$ and $(4,-2)$ is $\frac{-2-0}{4-0}=\frac{-2}{4}=-\frac{1}{2}$

Hence, the required slope of the line is $-\frac{1}{2}$.

Question 6:

Without using the Pythagoras theorem, show that the points $(4,4),(3,5)$ and $(-1,-1)$ are the vertices of a right angled triangle.

Answer 6:

The vertices of the given triangle are $A(4,4), B(3,5)$, and $C(-1,-1)$.

It is known that the slope (m) of a non-vertical line passing through the points (x_1, y_1)

and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$.

Slope of $\text{AB}(m_1) = \frac{5-4}{3-4} = -1$

Slope of $\text{BC}(m_2) = \frac{-1-5}{-1-3} = \frac{-6}{-4} = \frac{3}{2}$

Slope of CA $(m_3) = \frac{4+1}{4+1} = \frac{5}{5} = 1$

It is observed that $m_1 m_3 = -1$

This shows that line segments A B and C A are perpendicular to each other

And the given triangle is right angled at A(4,4).

Thus, the points $(4,4), (3,5)$, and $(-1,-1)$ are the vertices of a right-angled triangle.

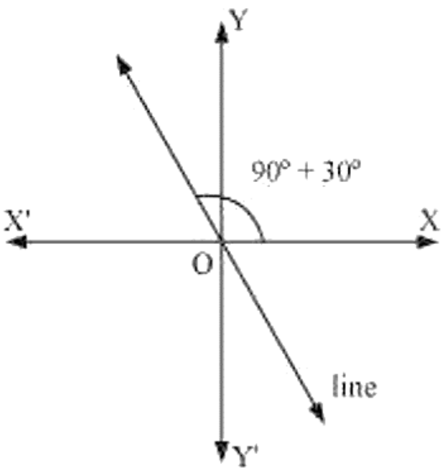
Question 7:

Find the slope of the line, which makes an angle of 30° with the positive direction of y axis measured anticlockwise.

Answer 7:

If a line makes an angle of 30° with the positive direction of the y-axis measured anticlockwise, then the angle made by the line with the positive direction of the x-axis

measured anticlockwise is $90^\circ + 30^\circ = 120^\circ$



Thus, the slope of the given line is $\tan 120^\circ = \tan(180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$

Question 8:

Find the value of x for which the points $(x,-1),(2,1)$ and $(4,5)$ are collinear.

Answer 8:

If points $A(x,-1), B(2,1)$, and $C(4,5)$ are collinear, then

Slope of $AB =$ Slope of B C

$$\Rightarrow \frac{1-(-1)}{2-x}=\frac{5-1}{4-2}$$

$$\Rightarrow \frac{1+1}{2-x}=\frac{4}{2}$$

$$\Rightarrow \frac{2}{2-x}=2$$

$$\Rightarrow 2=4-2x$$

$$\Rightarrow 2x=2$$

$$\Rightarrow x=1$$

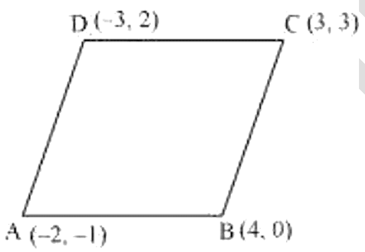
Thus, the required value of x is 1 .

Question 9:

Without using distance formula, show that points $(-2,-1),(4,0),(3,3)$ and $(-3,2)$ are vertices of a parallelogram.

Answer 9:

Let points $(-2,-1),(4,0),(3,3)$, and $(-3,2)$ be respectively denoted by \$A, B, C\$, and D.



Slope of $AB=\frac{0+1}{4+2}=\frac{1}{6}$

Slope of $CD=\frac{2-3}{-3-3}=\frac{-1}{-6}=\frac{1}{6}$

$\Rightarrow$ Slope of $AB =$ Slope of C D

$\Rightarrow AB$ and C D are parallel to each other.

Now, slope of $BC=\frac{3-0}{3-4}=\frac{3}{-1}=-3$

Slope of $AD=\frac{2+1}{-3+2}=\frac{3}{-1}=-3$

$\Rightarrow$ Slope of $AB=$ Slope of C D

$\Rightarrow AB$ and C D are parallel to each other.

Now, slope of $BC=\frac{3-0}{3-4}=\frac{3}{-1}=-3$

Slope of $AD=\frac{2+1}{-3+2}=\frac{3}{-1}=-3$

$\Rightarrow$ Slope of $BC=$ Slope of A D

$\Rightarrow BC$ and A D are parallel to each other.

Therefore, both pairs of opposite sides of quadrilateral ABCD are parallel. Hence, ABCD is a parallelogram.

Thus, points $(-2,-1),(4,0),(3,3)$, and $(-3,2)$ are the vertices of a parallelogram.

Question 10:

Find the angle between the x-axis and the line joining the points $(3,-1)$ and $(4,-2)$.

Answer 10:

The slope of the line joining the points $(3,-1)$ and $(4,-2)$ is $m=\frac{-2-(-1)}{4-3}=-2+1=-1$

Now, the inclination (θ) of the line joining the points $(3,-1)$ and $(4,-2)$ is given by $\tan\theta=-1$

$\Rightarrow \theta=\left(90^\circ+45^\circ\right)=135^\circ$

Thus, the angle between the x-axis and the line joining the points $(3,-1)$ and $(4,-2)$ is 135°.

Question 11:

The slope of a line is double of the slope of another line. If tangent of the angle between them is $1/3$, find the slopes of the lines.

Answer 11:

Let m_1 and m be the slopes of the two given lines such that $m_1=2m$.

We know that if θ is the angle between the lines I_1 and I_2 with slopes m_1 and m_2, then

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$$

It is given that the tangent of the angle between the two lines is $\frac{1}{3}$.

$$\therefore \frac{1}{3}=\left|\frac{m-2m}{1+(2m)\cdot m}\right|$$

$$\Rightarrow \frac{1}{3}=\left|\frac{-m}{1+2m^2}\right|$$

$$\Rightarrow \frac{1}{3}=\frac{-m}{1+2m^2} \text{ or } \frac{1}{3}=-\left(\frac{-m}{1+2m^2}\right)=\frac{m}{1+2m^2}$$

Case 1

$$\Rightarrow \frac{1}{3}=\frac{-m}{1+2m^2}$$

$$\Rightarrow 1+2m^2=-3m$$

$$\Rightarrow 2m^2+3m+1=0$$

$$\Rightarrow 2m^2+2m+m+1=0$$

$$\Rightarrow 2m(m+1)+1(m+1)=0$$

$$\Rightarrow (m+1)(2m+1)=0$$

$$\Rightarrow m=-1 \text{ or } m=-\frac{1}{2}$$

If $m=-1$, then the slopes of the lines are -1 and -2.

If $m=-\frac{1}{2}$, then the slopes of the lines are $-\frac{1}{2}$ and -1.

Case 2

$$\frac{1}{3}=\frac{m}{1+2m^2}$$

$$\Rightarrow 2m^2+1=3m$$

$$\Rightarrow 2m^2-3m+1=0$$

$$\Rightarrow 2m^2-2m-m+1=0$$

$$\Rightarrow 2m(m-1)-1(m-1)=0$$

$$\Rightarrow (m-1)(2m-1)=0$$

$$\Rightarrow m=1 \text{ or } m=\frac{1}{2}$$

If $m=1$, then the slopes of the lines are 1 and 2 .

If $m=\frac{1}{2}$, then the slopes of the lines are $\frac{1}{2}$ and 1 Hence, the slopes of the lines are -1 and -2 or $-\frac{1}{2}$ and -1 or 1 and 2 or $\frac{1}{2}$ and 1 .

Question 12:

A line passes through $\left(x_1, y_1\right)$ and (h, k). If slope of the line is m, show that $k-y_1=m\left(h-x_1\right)$

Answer 12:

The slope of the line passing through $\left(x_1, y_1\right)$ and (h, k) is $\frac{k-y_1}{h-x_1}$.

It is given that the slope of the line is m.

$\therefore \frac{k-y_1}{h-x_1}=m$

$\Rightarrow k-y_1=m\left(h-x_1\right)$

Hence, $k-y_1=m\left(h-x_1\right)$

Question 13:

If three point $(h, 0),(a, b)$ and $(0, k)$ lie on a line, show that $\frac{a}{h}+\frac{b}{k}=1$.

Answer 13:

If the points $A(h, 0), B(a, b)$, and $C(0, k)$ lie on a line, then

Slope of $AB=$ Slone of B C

$\frac{b-0}{a-h}=\frac{k-b}{0-a}$

$\Rightarrow \frac{b}{a-h}=\frac{k-b}{-a}$

$\Rightarrow-ab=(k-b)(a-h)$

$\Rightarrow-ab=ka-kh-ab+bh$

$\Rightarrow ka+bh=kh$

On dividing both sides by k h, we obtain

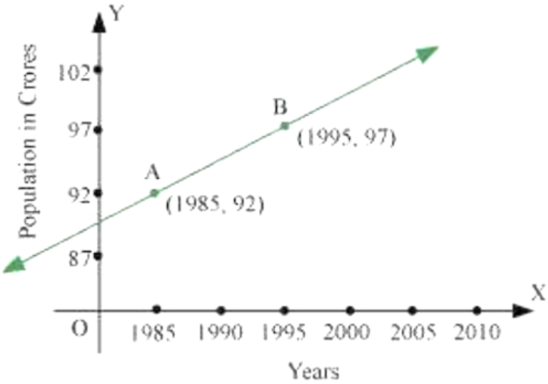
$$\frac{ka}{kh}+\frac{bh}{kh}=\frac{kh}{kh}$$

$$\Rightarrow \frac{a}{h}+\frac{b}{k}=1$$

Hence, $\frac{a}{h}+\frac{b}{k}=1$

Question 14:

Consider the given population and year graph. Find the slope of the line A B and using it, find what will be the population in the year 2010 ?



Answer 14:

Since line AB passes through points A $(1985,92)$ and B $(1995,97)$, its slope is

$$\frac{97-92}{1995-1985}=\frac{5}{10}=\frac{1}{2}$$

Let y be the population in the year 2010 . Then, according to the given graph, line A B must pass through point C $(2010, y)$.

$\therefore$ Slope of $AB=$ Slope of B C

$$\Rightarrow \frac{1}{2}=\frac{y-97}{2010-1995}$$

$$\Rightarrow \frac{1}{2}=\frac{y-97}{15}$$

$$\Rightarrow \frac{15}{2}=y-97$$

$$\Rightarrow y-97=7.5$$

$$\Rightarrow y=7.5+97=104.5$$

Thus, the slope of line A B is $\frac{1}{2}$ while in the year 2010, the population will be 104.5 crores.

Example 6

Find the equations of the lines parallel to axes and passing through $(-2,3)$

Solution

The general expression is

When a line parallel to axes and passing through a point

It's equation will be in the form $y=mx+c$

Here,

The y-coordinate of every point on the line parallel to x-axis is 3,

therefore,

The equation of the line parallel to x-axis and passing through $(-2,3)$ is $y=3$.

Similarly,

The equation of the line parallel to y-axis and passing through $(-2,3)$ is $x=-2$.

Example 7

Find the equation of the line through $(-2,3)$ with slope -4.

Solution

The general expression for equation of a line is

$y=mx+c$

Where m is the slope of the line

Here $m=-4$ and given point (x_0, y_0) is $(-2,3)$.

By slope-intercept form formula

equation of the given line is

$y=mx+c$

$y-3=-4(x+2)$ or

$4x+y+5=0$,

which is the required equation.

Example 8

Write the equation of the line through the points $(1,-1)$ and $(3,5)$.

Solution

Here, the points are given.

Let the points be $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$

Slope of the line joining the two points will be the slope of line whose equation we have to find.

Slope of the line joining two points $=\dfrac{y_2-y_1}{x_2-x_1}=m$

Standard form of Eq. of the line is $\rightarrow$

$y=mx+c$ where $m=$ slope of the line

Eq. of the line is

$y=mx+c$

$$\left(y-y_1\right)=\frac{y_2-y_1}{x_2-x_1}\left(x-x_1\right)$$

.After substituting the values if given and simplifying it, we will obtain the Eq. Of line in the stand form i.e.,

$y=mx+c$

Here $x_1=1, y_1=-1, x_2=3$ and $y_2=5$.

Using two-point form , above for the equation of the line,

we have

$$y-(-1)=\frac{5-(-1)}{3-1}(x-1)$$

or

$-3x+y+4=0$, which is the required equation.

Example 9

Write the equation of the lines for which $\tan\theta=\dfrac{1}{2}$, where θ is the inclination of the line and (i) y -intercept is $-\dfrac{3}{2}$ (ii) x -intercept is 4.

Solution

(i) Here, slope of the line is $m=\tan\theta=\frac{1}{2}$ and y - intercept $c=-\frac{3}{2}$.

Therefore, by slope-intercept form

$y=mx+c$

Here the m represents the slope of the line

Then substitute the values in the equation

We get,

$y=\frac{1}{2}x-\frac{3}{2}$ or $2y-x+3=0$

(ii) Here, we have $m=\tan\theta=\frac{1}{2}$ and $d=4$.

The general expression for equation of a line is

$y=mx+c$

Therefore,

by slope-intercept form above, the equation of the line is

$y=\frac{1}{2}(x-4)$ or $2y-x+4=0$

which is the required equation.

Example 10

Find the equation of the line, which makes intercepts -3 and 2 on the x - and y -axes respectively.

Solution

Here

$a=-3$ and $b=2$.

The general expression for equation of a line is

$y=mx+c$

By the intercept form,

The equation of the line is

Substitute the values in the intercept form,

Then

$$\frac{x}{-3}+\frac{y}{2}=1$$

Cross multiply the terms

Which again solving we get

$$2x-3y+6=0$$

Which is the required relation

Example 11

Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

Solution

Here, we are given $p=4$ and $\omega=15^{0}$

Now $\cos 15^{\circ}=\frac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15^{\circ}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

By the intercept form, the equation is

$$x\cos 15^{0}+y\sin 15^{\circ}=4$$

$$x\cos\left(15^{\circ}\right)=4-\frac{\sqrt{2-\sqrt{3}}}{2}y$$

Simplify $\cos\left(15^{\circ}\right):\frac{\sqrt{2+\sqrt{3}}}{2}$

$$x\frac{\sqrt{2+\sqrt{3}}}{2}=4-\frac{\sqrt{2-\sqrt{3}}}{2}y$$

Multiply both sides by 2

$$2x\frac{\sqrt{2+\sqrt{3}}}{2}=2\cdot 4-2\cdot\frac{\sqrt{2-\sqrt{3}}}{2}y$$

Simplify

$$\sqrt{2+\sqrt{3}}x=8-\sqrt{2-\sqrt{3}}y$$

Divide both sides by $\sqrt{2+\sqrt{3}}$

$$\frac{\sqrt{2+\sqrt{3}}x}{\sqrt{2+\sqrt{3}}}=\frac{8}{\sqrt{2+\sqrt{3}}}-\frac{\sqrt{2-\sqrt{3}}y}{\sqrt{2+\sqrt{3}}}$$

Simplify

$$x=(2-\sqrt{3})\sqrt{2+\sqrt{3}}(8-\sqrt{2-\sqrt{3}}y)$$

This is the required equation.

Example 12

The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that $\mathrm{K}=273$ when $\mathrm{F}=32$ and that $\mathrm{K}=373$ when $\mathrm{F}=212$. Express K in terms of F and find the value of F, when $K=0$.

Solution

Assuming F along x-axis and K along y-axis,

we have two points $(32,273)$ and $(212,373)$ in XY-plane.

By two-point form, the point (F,K) satisfies the equation

$$\mathrm{K}-273=\frac{373-273}{212-32}(\mathrm{F}-32) \text{ or } \mathrm{K}-273=\frac{100}{180}(\mathrm{F}-32)$$

Simplify

$$\frac{373-273}{212-32}(F-32): \quad \frac{5(F-32)}{9}$$

$$K-273=\frac{5(F-32)}{9}$$

It can also be written as

$$\mathrm{K}=\frac{5}{9}(\mathrm{F}-32)+273$$

which is the required relation.

When $\mathrm{K}=0$, Equation (1) gives

$$0=\frac{5}{9}(\mathrm{F}-32)+273 \quad \text{or} \quad \mathrm{F}-32=-\frac{273\times 9}{5}=-491.4 \quad \text{or} \quad \mathrm{F}=-459.4$$

Exercise 10.2

Question 1:

Write the equations for the x and y-axes.

Answer 1:

The y-coordinate of every point on the x-axis is 0.

Therefore, the equation of the x-axis is $y=0$.

The x-coordinate of every point on the y-axis is 0.

Therefore, the equation of the y-axis is $y=0$.

Question 2:

Find the equation of the line which passes through the point $(-4,3)$ with slope $\frac{1}{2}$.

Answer 2:

We know that the equation of the line passing through point (x_0, y_0), whose slope is m, is

$$(y-y_0)=m(x-x_0)$$

Thus, the equation of the line passing through point $(-4,3)$, whose slope is $\frac{1}{2}$, is $(y-3)=\frac{1}{2}(x+4)$

$$2(y-3)=x+4$$

$$2y-6=x+4$$

i.e., $x-2y+10=0$

Question 3:

Find the equation of the line which passes though $(0,0)$ with slope m.

Answer 3:

We know that the equation of the line passing through point (x_0, y_0), whose slope is m, is

$$(y-y_0)=m(x-x_0)$$

Thus, the equation of the line passing through point $(0,0)$, whose slope is m, is

$$(y-0)=m(x-0)$$

i.e., $y = mx$

Question 4:

Find the equation of the line which passes though $(2, 2\sqrt{3})$ and is inclined with the x-axis at an angle of 75°

Answer 4:

The slope of the line that inclines with the x-axis at an angle of 75° is $m = \tan 75^\circ$

$$\Rightarrow m = \tan\left(45^\circ + 30^\circ\right) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \cdot \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

We know that the equation of the line passing through point (x_0, y_0)

whose slope is m, is

$$(y - y_0) = m(x - x_0)$$

Thus, if a line passes though $(2, 2\sqrt{3})$ and inclines with the x-axis at an angle of 75°,

then the equation of the line is given as

$$(y - 2\sqrt{3}) = \frac{\sqrt{3}+1}{\sqrt{3}-1}(x - 2)$$

$$(y - 2\sqrt{3})(\sqrt{3} - 1) = (\sqrt{3} + 1)(x - 2)$$

$$y(\sqrt{3} - 1) - 2\sqrt{3}(\sqrt{3} - 1) = x(\sqrt{3} + 1) - 2(\sqrt{3} + 1)$$

$$(\sqrt{3} + 1)x - (\sqrt{3} - 1)y = 2\sqrt{3} + 2 - 6 + 2\sqrt{3}$$

$$(\sqrt{3} + 1)x - (\sqrt{3} - 1)y = 4\sqrt{3} - 4$$

i.e., $(\sqrt{3} + 1)x - (\sqrt{3} - 1)y = 4(\sqrt{3} - 1)$

Question 5:

Find the equation of the line which intersects the x-axis at a distance of 3 units to the left of origin with slope -2.

Answer 5:

It is known that if a line with slope m makes x-intercept d, then the equation of the line is given as $y = m(x - d)$

For the line intersecting the x-axis at a distance of 3 units to the left of the origin, $d=-3$

The slope of the line is given as $m=-2$

Thus,

the required equation of the given line is $y=-2[x-(-3)]y=-2x-6$

i.e., $2x+y+6=0$

Question 6:

Find the equation of the line which intersects the y-axis at a distance of 2 units above the origin and makes an angle of 30° with the positive direction of the x-axis.

Answer 6:

It is known that if a line with slope m makes y-intercept c,

then the equation of the line is given as $y=mx+c$

Here, $c=2$ and $m=\tan 30^{\circ}=\frac{1}{\sqrt{3}}$

Thus,

the required equation of the given line is

$$y=\frac{1}{\sqrt{3}}x+2$$

$$y=\frac{x+2\sqrt{3}}{\sqrt{3}}$$

$$\sqrt{3}y=x+2\sqrt{3}$$

i.e., $x-\sqrt{3}y+2\sqrt{3}=0$

Question 7:

Find the equation of the line which passes through the points $(-1,1)$ and $(2,-4)$.

Answer 7:

It is known that the equation of the line passing through points (x_1, y_1) and (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Therefore,

the equation of the line passing through the points $(-1,1)$ and $(2,-4)$ is

$$(y-1)=\frac{-4-1}{2+1}(x+1)$$

$$(y-1)=\frac{-5}{3}(x+1)$$

$$3(y-1)=-5(x+1)$$

$$3y-3=-5x-5$$

i.e., $5x+3y+2=0$

Question 8:

Find the equation of the line which is at a perpendicular distance of 5 units from the origin and the angle made by the perpendicular with the positive x-axis is 30°

Answer 8:

If p is the length of the normal from the origin to a line and ω is the angle made by the normal with the positive direction of the x-axis,

then the equation of the line is given by

$$x\cos\omega+y\sin\omega=p$$

Here, $p=5$ units and $\omega=30^{\circ}$

Thus,

the required equation of the given line is x

$$\cos 30^{\circ}+y\sin 30^{\circ}=5$$

$$x\frac{\sqrt{3}}{2}+y\cdot\frac{1}{2}=5$$

i.e., $\sqrt{3}x+y=10$

Question 9:

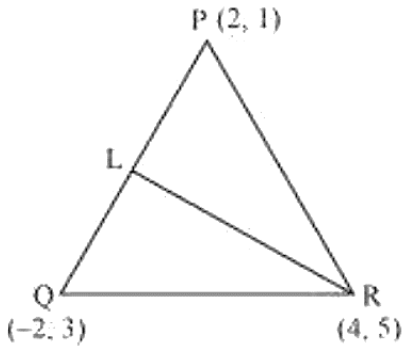
The vertices of $\triangle PQR$ are $P(2,1), Q(-2,3)$ and $R(4,5)$. Find equation of the median through the vertex R.

Answer 9:

It is given that the vertices of $\triangle PQR$ are $P(2,1), Q(-2,3)$, and $R(4,5)$.

Let RL be the median through vertex R.

Accordingly, L is the mid-point of PQ. By mid-point formula, the coordinates of point L are given by $\left(\frac{2-2}{2}, \frac{1+3}{2}\right) = (0, 2)$



It is known that the equation of the line passing through points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Therefore, the equation of RL can be determined by substituting $(x_1, y_1) = (4, 5)$

and $(x_2,, y_2) = (0, 2)$

Hence, $y - 5 = \frac{2-5}{0-4}(x-4)$

$\Rightarrow y - 5 = \frac{-3}{-4}(x-4)$

$\Rightarrow 4(y-5) = 3(x-4)$

$\Rightarrow 4y - 20 = 3x - 12$

$\Rightarrow 3x - 4y + 8 = 0$

Thus, the required equation of the median through vertex R is $3x - 4y + 8 = 0$.

Question 10:

Find the equation of the line passing through $(-3, 5)$ and perpendicular to the line through the points $(2, 5)$ and $(-3, 6)$.

Answer 10:

The slope of the line joining the points $(2, 5)$ and $(-3, 6)$ is $m = \frac{6-5}{-3-2} = \frac{1}{-5}$

We know that two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other.

Therefore, slope of the line perpendicular to the line through the points

$(2,5)$ and $(-3,6)$ $= -\frac{1}{m} = -\frac{1}{\left(\frac{-1}{5}\right)} = 5$

Now, the equation of the line passing through point $(-3,5)$, whose slope is 5, is

$(y-5) = 5(x+3)$

$y-5 = 5x+15$

i.e., $5x - y + 20 = 0$

Question 11:

A line perpendicular to the line segment joining the points $(1,0)$ and $(2,3)$ divides it in the ratio 1: n. Find the equation of the line.

Answer 11:

According to the section formula, the coordinates of the point that divides the line segment joining the points $(1,0)$ and $(2,3)$ in the ratio 1: n is given by $\left(\frac{n(1)+1(2)}{1+n}, \frac{n(0)+1(3)}{1+n}\right) = \left(\frac{n+2}{n+1}, \frac{3}{n+1}\right)$

The slope of the line joining the points $(1,0)$ and $(2,3)$ is $m = \frac{3-0}{2-1} = 3$

We know that two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other.

Therefore, slope of the line that is perpendicular to the line joining the points

$(1,0)$ and $(2,3) = -\frac{1}{m} = -\frac{1}{3}$

Now, the equation of the line passing through $\left(\frac{n+2}{n+1}, \frac{3}{n+1}\right)$ and whose slope is $-\frac{1}{3}$ is given by

$\left(y - \frac{3}{n+1}\right) = \frac{-1}{3}\left(x - \frac{n+2}{n+1}\right)$

$\Rightarrow 3[(n+1)y - 3] = -[x(n+1) - (n+2)]$

$\Rightarrow 3(n+1)y - 9 = -(n+1)x + n + 2$

$\Rightarrow (1+n)x + 3(1+n)y = n + 11$

Question 12:

Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $(2,3)$.

Answer 12:

The equation of a line in the intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

Here, a and b are the intercepts on x and y axes respectively.

It is given that the line cuts off equal intercepts on both the axes.

This means that $a=b$.

Accordingly, equation (i) reduces to

$$\frac{x}{a}+\frac{y}{a}=1$$

$$\Rightarrow x+y=a$$

Since the given line passes through point $(2,3)$, equation (ii) reduces to

$$2+3=a \Rightarrow a=5$$

On substituting the value of a in the equation(ii)

We obtain

$x+y=5$, which is the required equation of the line

Question 13:

Find equation of the line passing through the point $(2,2)$ and cutting off intercepts on the axes whose sum is 9.

Answer 13:

The equation of a line in the intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

Here, a and b are the intercepts on x and y axes respectively.

It is given that $a+b=9 \Rightarrow b=9-a \ldots$ (ii) From equations (i) and (ii), we obtain

$$\frac{x}{a}+\frac{y}{9-a}=1$$

It is given that the line passes through point $(2,2)$. Therefore, equation (iii) reduces to

$$\frac{2}{a}+\frac{2}{9-a}=1$$

$$\Rightarrow 2\left(\frac{1}{a}+\frac{1}{9-a}\right)=1$$

$$\Rightarrow 2\left(\frac{9-a+a}{a(9-a)}\right)=1$$

$$\Rightarrow \frac{18}{9a-a^2}=1$$

$$\Rightarrow 18=9a-a^2$$

$$\Rightarrow a^2-9a+18=0$$

$$\Rightarrow a^2-6a-3a+18=0$$

$$\Rightarrow a(a-6)-3(a-6)=0$$

$$\Rightarrow (a-6)(a-3)=0$$

$$\Rightarrow a=6 \text{ or } a=3$$

If $a=6$ and $b=9-6=3$, then the equation of the line is

$$\frac{x}{6}+\frac{y}{3}=1 \Rightarrow x+2y-6=0$$

If $a=3$ and $b=9-3=6$, then the equation of the line is $\frac{x}{3}+\frac{y}{6}=1 \Rightarrow 2x+y-6=0$

Question 14:

Find equation of the line through the point $(0,2)$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

Answer 14:

The slope of the line making an angle $\frac{2\pi}{3}$ with the positive x-axis is $m=\tan\left(\frac{2\pi}{3}\right)=-\sqrt{3}$

Now, the equation of the line passing through point $(0,2)$ and having a slope $-\sqrt{3}$ is

$$(y-2)=-\sqrt{3}(x-0)$$

$$y-2=-\sqrt{3}x$$

i.e., $\sqrt{3}x+y-2=0$

The slope of line parallel $\sqrt{3}x+y-2=0$ is $-\sqrt{3}$ to line.

It is given that the line parallel to line $\sqrt{3}x+y-2=0$ crosses the y-axis 2 units below the origin i.e., it passes through point $(0,-2)$.

Hence, the equation of the line passing through point $(0,-2)$ and having a slope $-\sqrt{3}$ is

$$y-(-2)=-\sqrt{3}(x-0)$$

$$y+2=-\sqrt{3}x$$

$$\sqrt{3}x+y+2=0$$

Question 15:

The perpendicular from the origin to a line meets it at the point $(-2,9)$, find the equation of the line.

Answer 15:

The slope of the line joining the origin $(0,0)$ and point $(-2,9)$ is $m_1=\frac{9-0}{-2-0}=-\frac{9}{2}$

Accordingly, the slope of the line perpendicular to the line joining the origin and point $(-2,9)$ is

$$m_2=-\frac{1}{m_1}=-\frac{1}{\left(-\frac{9}{2}\right)}=\frac{2}{9}$$

Now, the equation of the line passing through point $(-2,9)$ and having a slope m_2 is

$$(y-9)=\frac{2}{9}(x+2)$$

$$9y-81=2x+4$$

i.e., $2x-9y+85=0$

Question 16:

The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $\mathrm{L}=124.942$ when $\mathrm{C}=20$ and $\mathrm{L}=125.134$ when $\mathrm{C}=110$, express L in terms of

C.

Answer 16:

It is given that when $C=20$, the value of L is \$124.942\$, whereas when $C=110$, the value of L is 125.134

Accordingly, points $(20,124.942)$ and $(110,125.134)$ satisfy the linear relation between L and C.

Now, assuming C along the x-axis and L along the y-axis, we have two points

i.e., $(20, 124.942)$ *and* (110,125.134) in the X Y plane.

Therefore, the linear relation between L and C is the equation of the line passing through points $(20,124.942)$ and $(110,125.134)$.

$$(L-124.942)=\frac{125.134-124.942}{110-20}(C-20)$$

$$L-124.942=\frac{0.192}{90}(C-20)$$

i.e., $L=\frac{0.192}{90}(C-20)+124.942$, which is the required linear relation

Question 17:

The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs $14/$ litre and 1220 litres of milk each week at Rs $16/$ litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs $17/$ litre?

Answer 17:

The relationship between selling price and demand is linear.

Assuming selling price per litre along the x-axis and demand along the y-axis, we have two points i.e., $(14,980)$ and $(16,1220)$ in the X Y plane that satisfy the linear relationship between selling price and demand.

Therefore, the linear relationship between selling price per litre and demand is the equation of the line passing through points $(14,980)$ and $(16,1220)$.

$$y-980=\frac{1220-980}{16-14}(x-14)$$

$$y-980=\frac{240}{2}(x-14)$$

$y-980=120(x-14)$

i.e., $y=120(x-14)+980$

When $x=$ Rs $17/$ litre,

$y=120(17-14)+980$

$\Rightarrow y=120\times 3+980=360+980=1340$

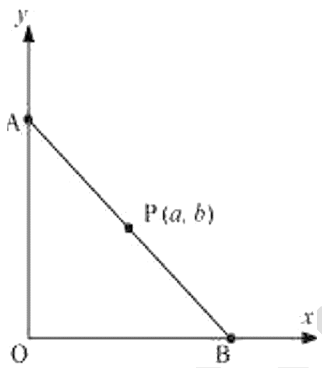
Thus, the owner of the milk store could sell 1340 litres of milk weekly at Rs $17/$ litre.

Question 18:

$\mathrm{P}(a,b)$ is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$

Answer 18:

Let A B be the line segment between the axes and let $P(a,b)$ be its mid-point.



Let the coordinates of A and B be $(0,y)$ and $(x,0)$ respectively.

Since P (a,b) is the mid-point of AB,

$\left(\frac{0+x}{2},\frac{y+0}{2}\right)=(a,b)$

$\Rightarrow\left(\frac{x}{2},\frac{y}{2}\right)=(a,b)$

$\Rightarrow\frac{x}{2}=a$ and $\frac{y}{2}=b$

$\therefore x=2a$ and $y=2b$

Thus, the respective coordinates of A and B are $(0,2b)$ and $(2a,0)$.

The equation of the line passing through points $(0,2b)$ and $(2a,0)$ is

$$(y-2b)=\frac{(0-2b)}{(2a-0)}(x-0)$$

$$y-2b=\frac{-2b}{2a}(x)$$

$$a(y-2b)=-bx$$

$$ay-2ab=-bx$$

i.e., $bx+ay=2ab$

On dividing both sides by a b, we obtain

$$\frac{bx}{ab}+\frac{ay}{ab}=\frac{2ab}{ab}$$

$$\Rightarrow\frac{x}{a}+\frac{y}{b}=2$$

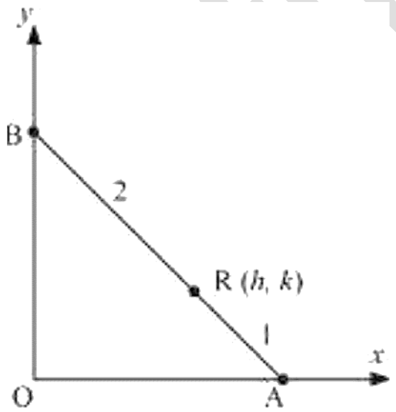
Thus, the equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.

Question 19:

Point R (h,k) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

Answer 19:

Let A B be the line segment between the axes such that point $R(h,k)$ divides A B in the ratio 1: 2 .



Let the respective coordinates of A and B be $(x,0)$ and $(0,y)$.

Since point $R(h,k)$ divides AB in the ratio 1: 2, according to the section formula,

$$(h,k)=\left(\frac{1\times 0+2\times x}{1+2},\frac{1\times y+2\times 0}{1+2}\right)$$

$$\Rightarrow (h,k)=\left(\frac{2x}{3},\frac{y}{3}\right)$$

$$\Rightarrow h=\frac{2x}{3} \text{ and } k=\frac{y}{3}$$

$$\Rightarrow x=\frac{3h}{2} \text{ and } y=3k$$

Therefore, the respective coordinates of A and B are $\left(\frac{3h}{2},0\right)$ and $(0,3k)$

Now, the equation of line AB passing through points $\left(\frac{3h}{2},0\right)$ and $(0,3k)$ is

$$(y-0)=\frac{3k-0}{0-\frac{3h}{2}}\left(x-\frac{3h}{2}\right) \quad y=-\frac{2k}{h}\left(x-\frac{3h}{2}\right) \quad hy=-2kx+3hk$$

i.e., $2kx+hy=3hk$

Thus,

the required equation of the line is $2kx+hy=3hk$

Question 20:

By using the concept of equation of a line, prove that the three points $(3,0),(-2,-2)$ and $(8,2)$ are collinear.

Answer 20:

In order to show that points $(3,0),(-2,-2)$, and $(8,2)$ are collinear,

it suffices to show that the line passing through points $(3,0)$ and $(-2,-2)$ also passes through point $(8,2)$.

The equation of the line passing through points $(3,0)$ and $(-2,-2)$ is

$$(y-0)=\frac{(-2-0)}{(-2-3)}(x-3)$$

$$y=\frac{-2}{-5}(x-3)$$

$$5y=2x-6$$

i.e., $2x-5y=6$

It is observed that at $x=8$ and $y=2$,

L.H.S. $= 2\times 8 - 5\times 2 = 16 - 10 = 6 =$ R.H.S.

Therefore,

the line passing through points $(3,0)$ and $(-2,-2)$ also passes through point $(8,2)$. Hence, points $(3,0),(-2,-2)$, and $(8,2)$ are collinear.

Example 13

Equation of a line is $3x-4y+10=0$. Find its (i) slope, (ii) $x-$ and y-intercepts.

Solution

(i) Given equation $3x-4y+10=0$ can be written as

$$y=\frac{3}{4}x+\frac{5}{2} \quad \ldots\ldots\ldots(1)$$

The general expression for equation of a line is

$$y=mx+c$$

Compare the equation (1) with the given equation

So ,

we have slope of the given line as $m=\frac{3}{4}$.

(ii) Equation $3x-4y+10=0$ can be written as $3x-4y=-10$

$$\frac{x}{-\frac{10}{3}}+\frac{y}{\frac{5}{2}}=1 \ldots\ldots\ldots.(2)$$

Comparing (2) with $\frac{x}{a}+\frac{y}{b}=1$,

we have x-intercept as $a=-\frac{10}{3}$

and y-intercept as $b=\frac{5}{2}$.

Example 14

Reduce the equation $\sqrt{3}x+y-8=0$ into normal form. Find the values of p and ω.

Solution

Given equation is

$$\sqrt{3}x+y-8=0 \quad ....(1)$$

Dividing (1) by $\sqrt{(\sqrt{3})^2+(1)^2}=2$, we get

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y-\frac{1}{2}y=4-\frac{1}{2}y$$

Simplify

$$\frac{\sqrt{3}}{2}x=4-\frac{1}{2}y$$

Multiply both sides by 2

$$2\cdot\frac{\sqrt{3}}{2}x=2\cdot 4-2\cdot\frac{1}{2}y$$

Simplify

$$\sqrt{3}x=8-y$$

Divide both sides by $\sqrt{3}$

$$\frac{\sqrt{3}x}{\sqrt{3}}=\frac{8}{\sqrt{3}}-\frac{y}{\sqrt{3}}$$

Simplify

$$x=\frac{\sqrt{3}(8-y)}{3}$$

$$\cos 30^{\circ}x+\sin 30^{\circ}y=4 \quad ....(2)$$

Comparing (2) *with* $x\cos\omega+y\sin\omega=p$,

we get $p=4$ and $\omega=30^{\circ}$.

Example15

Find the angle between the lines $y-\sqrt{3}x-5=0$ and $\sqrt{3}y-x+6=0$.

Solution

Given lines are

$$y-\sqrt{3}x-5=0 \text{ or } y=\sqrt{3}x+5$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Putting the values of m_1 and m_2 in (3), we get

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}}\right|$$

$$= \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1+1}\right|$$

Join $\frac{1}{\sqrt{3}} - \sqrt{3}: \quad -\frac{2}{\sqrt{3}}$

$$= \left|\frac{-\frac{2}{\sqrt{3}}}{1+1}\right|$$

Add the numbers: $1+1=2$

$$= \left|\frac{-\frac{2}{\sqrt{3}}}{2}\right|$$

Simplify $\frac{-\frac{2}{\sqrt{3}}}{2}: \quad -\frac{2}{2\sqrt{3}}$

$$= \left|-\frac{2}{2\sqrt{3}}\right|$$

Divide the numbers: $\frac{2}{2}=1$

$$=\left|-\frac{1}{\sqrt{3}}\right|$$

Apply absolute rule: $|-a|=a$

$$=\frac{1}{\sqrt{3}}$$

which gives $\theta=30^\circ$. Hence, angle between two lines is either 30° or $180^\circ-30^\circ=150^\circ$.

Example 16

Show that two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where $b_1,b_2\neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1}=\frac{a_2}{b_2}$,

(ii) and (ii) Perpendicular if $a_1a_2+b_1b_2=0$.

Solution Given lines can be written as

$$y=-\frac{a_1}{b_1}x-\frac{c_1}{b_1}$$

$$y=-\frac{a_2}{b_2}x-\frac{c_2}{b_2}$$

Slopes of the lines (1) and (2) are $m_1=-\frac{a_1}{b_1}$ and $m_2=-\frac{a_2}{b_2}$, respectively.

Now

(i) Lines are parallel, if $m_1=m_2$, which gives

$$-\frac{a_1}{b_1}=-\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1}=\frac{a_2}{b_2}$$

(iii) The lines are perpendicular if

$$m_1\cdot m_2=-1,$$

which gives

$$\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}=-1 \text{ or } a_1a_2+b_1b_2=0$$

Example 17

Find the equation of a line perpendicular to the line $x-2y+3=0$ and passing through the point $(1,-2)$.

Solution

The general expression for equation of a line is

$y=mx+c$

Given line $x-2y-3=0$ can be written as

$$y=\frac{1}{2}x+\frac{3}{2}$$

The m represents the slope of the line

So the slope of the line is

$$m_1=\frac{1}{2}\ .$$

Therefore the slope of the line is perpendicular to line (1) is

$$m_2=-\frac{1}{m_1}=-2$$

Equation of the line with slope -2 and passing through the point $(1,-2)$ is

$y-(-2)=-2(x-1)$ or $y=-2x$

which is the required equation.

Example 18

Find the distance of the point $(3,-5)$ from the line $3x-4y-26=0$.

Solution

Given line is $3x-4y-26=0$

Comparing (1) with general equation of line $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$,

we get

$\mathrm{A}=3, \mathrm{B}=-4$ and $\mathrm{C}=-26$

Given point is $\left(x_1, y_1\right)=(3,-5)$.

The distance of the given point from given line is

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}=\frac{|3.3+(-4)(-5)-26|}{\sqrt{3^2+(-4)^2}}$$

Remove parentheses: $(-a)=-a,-(-a)=a$

$$=\frac{|3.3+4\cdot5-26|}{\sqrt{3^2+(-4)^2}}$$

$$|3.3+4\cdot5-26|=2.7$$

$$=\frac{2.7}{\sqrt{3^2+(-4)^2}}$$

$$=\frac{3}{5}$$

Example 19

Find the distance between the parallel lines $3x-4y+7=0$ and

$3x-4y+5=0$

Solution

Here $A=3,B=-4,C_1=7$

and $C_2=5.$

Therefore, the required distance is

$$d=\frac{|7-5|}{\sqrt{3^2+(-4)^2}}$$

$$|7-5|=2$$

$$d=\frac{2}{\sqrt{3^2+(-4)^2}}$$

Simplify

$$d=\frac{2}{5}$$

Exercise 10.3

Question 1:

Reduce the following equations into slope-intercept form and find their slopes and the y intercepts.

(i) $x+7y=0$

(ii) $6x+3y-5=0$

(iii) $y=0$

Answer 1:

(i) The given equation is $x+7y=0$. It can be written as

$$y=-\frac{1}{7}x+0$$

This equation is of the form $y=mx+c$, where $m=-\frac{1}{7}$ and $c=0$

Therefore, equation (1) is in the slope-intercept form, where the slope and the y intercept are $-\frac{1}{7}$ and 0 respectively.

(ii) The given equation is $6x+3y-5=0$

It can be written as

$$y=\frac{1}{3}(-6x+5)$$

$$y=-2x+\frac{5}{3}$$

This equation is of the form $y=mx+c$, where $m=-2$ and $c=\frac{5}{3}$.

Therefore, equation (2) is in the slope-intercept form, where the slope and the y intercept are -2 and $\frac{5}{3}$ respectively.

(iii) The given equation is $y=0$.

It can be written as

$y=0.x+0\ldots$ (3)

This equation is of the form $y=mx+c$,

where $m=0$ and $c=0$.

Therefore, equation (3) is in the slope-intercept form, where the slope and the y-intercept are 0 and 0 respectively.

Question 2:

Reduce the following equations into intercept form and find their intercepts on the axes.

(i) $3x+2y-12=0$

(ii) $4x-3y=6$

(iii) $3y+2=0$.

Answer 2:

(i) The given equation is $3x+2y-12=0$.

It can be written as

$3x+2y=12$ ……….(1)

$$\frac{3x}{12}+\frac{2y}{12}=1$$

i.e., $\frac{x}{4}+\frac{y}{6}=1$

This equation is of the form $\frac{x}{a}+\frac{y}{b}=1$

where $a=4$ and $b=6$

Therefore, equation (1) is in the intercept form, where the intercepts on the x and y axes are 4 and 6 respectively.

(ii) The given equation is $4x-3y=6$

It can be written as

$$\frac{4x}{6}-\frac{3y}{6}=1$$

$$\frac{2x}{3}-\frac{y}{2}=1$$

i.e., $\frac{x}{\left(\frac{3}{2}\right)}+\frac{y}{(-2)}=1$

…(2)

This equation is of the form $\frac{x}{a}+\frac{y}{b}=1$, where $a=\frac{3}{2}$ and $b=-2$.

Therefore, equation (2) is in the intercept form, where the intercepts on the x and y axes are $3/2$ and -2 respectively.

(iii) The given equation is $3y+2=0$.

It can be written as

$$3y=-2$$

i.e. $\dfrac{y}{\left(-\dfrac{2}{3}\right)}=1$

The equation is of the form $\dfrac{x}{a}+\dfrac{y}{b}=1$, where $a=0$ and $b=-\dfrac{2}{3}$

Therefore, equation (3) is in the intercept form, where the intercept on the y-axis $-\dfrac{2}{3}$ is and it has no intercept on the x-axis.

Question 3:

Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.

(i) $x-\sqrt{3}y+8=0$

(ii) $y-2=0$

(iii) $x-y=4$

Answer 3:

(i) The given equation is $x-\sqrt{3}y+8=0$.

It can be reduced as:

$$x-\sqrt{3}y=-8$$

$$\Rightarrow -x+\sqrt{3}y=8$$

On dividing both sides by $\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$, we obtain

$$-\frac{x}{2}+\frac{\sqrt{3}}{2}y=\frac{8}{2}$$

$$\Rightarrow \left(-\frac{1}{2}\right)x+\left(\frac{\sqrt{3}}{2}\right)y=4$$

$$\Rightarrow x\cos 120^\circ + y\sin 120^\circ = 4$$

Equation (1) is in the normal form.

On comparing equation (1) with the normal form of equation of line $x \cos\omega + y\sin\omega = p$, we obtain $\omega = 120^\circ$ and $p = 4$

Thus, the perpendicular distance of the line from the origin is 4, while the angle between the perpendicular and the positive x-axis is 120°.

(ii) The given equation is $y - 2 = 0$.

It can be reduced as $0 \cdot x + 1 \cdot y = 2$

On dividing both sides by $\sqrt{0^2 + 1^2} = 1$, we obtain $0.x + 1.y = 2$

$\Rightarrow x\cos 90^\circ + y\sin 90^\circ = 2 \ldots (1)$

$\cos\omega + y\sin\omega = p$, we obtain $\omega = 90^\circ$ and $p = 2$

Thus, the perpendicular distance of the line from the origin is 2, while the angle between the perpendicular and the positive x-axis is 90°.

(iii) The given equation is $x - y = 4$.

It can be reduced as $1 \cdot x + (-1)y = 4$

On dividing both sides by $\sqrt{1^2 + (-1)^2} = \sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}\mathrm{x} + \left(-\frac{1}{\sqrt{2}}\right)\mathrm{y} = \frac{4}{\sqrt{2}}$$

$$\Rightarrow \mathrm{x}\cos\left(2\pi - \frac{\pi}{4}\right) + \mathrm{y}\sin\left(2\pi - \frac{\pi}{4}\right) = 2\sqrt{2}$$

$$\Rightarrow \mathrm{x}\cos 315^\circ + \mathrm{y}\sin 315^\circ = 2\sqrt{2}$$

Equation (1) is in the normal form.

On comparing equation (1) with the normal form of equation of line $x\cos\omega + y\sin\omega = p$, we obtain $\omega = 315^\circ$ and $\mathrm{p} = 2\sqrt{2}$

Thus, the perpendicular distance of the line from the origin is $2\sqrt{2}$, while the angle between the perpendicular and the positive x-axis is 315°.

Question 4:

Find the distance of the point $(-1,1)$ from the line $12(x+6) = 5(y-2)$.

Answer 4:

The given equation of the line is $12(x+6) = 5(y-2)$.

$\Rightarrow 12x+72=5y-10$

$\Rightarrow 12x-5y+82=0\ldots$

On comparing equation (1) with general equation of line $Ax+By+C=0$, we obtain $A=12, B=-5$, and $C=82$

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1, y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$.

The given point is $(x_1, y_1)=(-1,1)$.

Therefore, the distance of point $(-1,1)$ from the given line

$$=\frac{|12(-1)+(-5)(1)+82|}{\sqrt{(12)^2+(-5)^2}} \text{ units } =\frac{|-12-5+82|}{\sqrt{169}} \text{ units } =\frac{|65|}{13} \text{ units } =5 \text{ units}$$

Question 5:

Find the points on the x-axis, whose distances from the line $\frac{x}{3}+\frac{y}{4}=1$ are 4 units.

Answer 5:

The given equation of line is

$$\frac{x}{3}+\frac{y}{4}=1$$

or, $4x+3y-12=0$

On comparing equation (1) with general equation of line $Ax+By+C=0$,

we obtain $A=4, B=3$, and $C=-12$

Let $(a,0)$ be the point on the x-axis whose distance from the given line is 4 units.

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point

(x_1, y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

Therefore,

$$4=\frac{|4a+3\times 0-12|}{\sqrt{4^2+3^2}}$$

$$\Rightarrow 4=\frac{|4a-12|}{5}$$

$$\Rightarrow |4a-12|=20$$

$$\Rightarrow \pm(4a-12)=20$$

$$\Rightarrow (4a-12)=20 \text{ or } -(4a-12)=20$$

$$\Rightarrow 4a=20+12 \text{ or } 4a=-20+12$$

$$\Rightarrow a=8 \text{ or } -2$$

Thus, the required points on the x-axis are $(-2,0)$ and $(8,0)$.

Question 6:

Find the distance between parallel lines

(i) $15x+8y-34=0$ and $15x+8y+31=0$

(ii) $I(x+y)+p=0$ and $I(x+y)-r=0$

Answer 6:

It is known that the distance (d) between parallel lines $Ax+By+C_1=0$ and $Ax+By+ C_2=0$ is given by

$$d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$$

Then

$15x+8y-34=0$ and $15x+8y+31=0$.

Here, $A=15, B=8, C_1=-34$, and $C_2=31$

Therefore, the distance between the parallel lines is

$$d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}=\frac{|-34-31|}{\sqrt{(15)^2+(8)^2}} \text{ units }=\frac{|-65|}{17} \text{ units }=\frac{65}{17} \text{ units}$$

(ii) The given parallel lines are $I(x+y)+p=0$ and $I(x+y)-r=0$

$Ix+ly+p=0$ and $Ix+ly-r=0$ Here,

$A=I, B=I, C_1=p$, and $C_2=-r$

Therefore, the distance between the parallel lines is

$$d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}=\frac{|p+r|}{\sqrt{I^2+l^2}} \text{ units } =\frac{|p+r|}{\sqrt{2l^2}} \text{ units } =\frac{|p+r|}{I\sqrt{2}} \text{ units } =\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right| \text{ units}$$

Question 7:

Find equation of the line parallel to the line $3x-4y+2=0$ and passing through the point $(-2,3)$.

Answer 7:

The equation of the given line is

$3x-4y+2=0$

or $y=\frac{3x}{4}+\frac{2}{4}$

or $y=\frac{3}{4}x+\frac{1}{2}$

which is of the form $y=mx+c$

$\therefore$ Slope of the given line $=3/4$

It is known that parallel lines have the same slope.

$\therefore$ Slope of the other line $=m=\frac{3}{4}$

Now, the equation of the line that has a slope of $\frac{3}{4}$ and passes through the point (-2,3)is

$(y-3)=\frac{3}{4}\{x-(-2)\}$

$4y-12=3x+6$

i.e., $3x-4y+18=0$

Question 8:

Find equation of the line perpendicular to the line $x-7y+5=0$ and having x intercept 3.

Answer 8:

The given equation of line is $x-7y+5=0$. Or, $y=\frac{1}{7}x+\frac{5}{7}$

which is of the form $y=mx+c$

$\therefore$ Slope of the given line $= \frac{1}{7}$

The slope of the line perpendicular to the line having a slope of $\frac{1}{7}$ is $m = -\frac{1}{\left(\frac{1}{7}\right)} = -7$

The equation of the line with slope -7 and x-intercept 3 is given by y

$\Rightarrow y = -7(x-3)$

$\Rightarrow y = -7x+21$

$\Rightarrow 7x+y=21$

Question 9:

Find angles between the lines $\sqrt{3}x+y=1$ and $x+\sqrt{3}y=1$

Answer 9:

The given lines are $\sqrt{3}x+y=1$ and $x+\sqrt{3}y=1$.

$y=-\sqrt{3}x+1 \quad \ldots(1) \qquad$ and $y=-\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}} \quad \ldots(2)$

The slope of line (1) is $m_1=-\sqrt{3}$,

while the slope of line (2) is $m_2=-\frac{1}{\sqrt{3}}$

The acute angle i.e., θ between the two lines is given by

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\tan\theta = \left|\frac{-\sqrt{3}+\frac{1}{\sqrt{3}}}{1+(-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right)}\right|$$

$$\tan\theta = \left|\frac{-3+1}{\sqrt{3}}\right|$$

$$\tan\theta = \frac{1}{\sqrt{3}}\left|\frac{-2}{2\times\sqrt{3}}\right|$$

$$\theta = 30^\circ$$

Thus, the angle between the given lines is either 30° or $180^\circ - 30^\circ = 150^\circ$.

Question 10:

The line through the points $(h,3)$ and $(4,1)$ intersects the line $7x-9y-19=0$, at right angle.

Find the value of h.

Answer 10:

The slope of the line passing through points $(h,3)$ and $(4,1)$ is

$$m_1 = \frac{1-3}{4-h} = \frac{-2}{4-h}$$

The slope of line $7x-9y-19=0$ or $y = \frac{7}{9}x - \frac{19}{9}$ is

$$m_2 = \frac{7}{9}$$

It is given that the two lines are perpendicular.

$$\therefore m_1 \times m_2 = -1$$

$$\Rightarrow \left(\frac{-2}{4-h}\right) \times \left(\frac{7}{9}\right) = -1$$

$$\Rightarrow \frac{-14}{36-9h} = -1$$

$$\Rightarrow 14 = 36 - 9h$$

$$\Rightarrow 9h = 36 - 14$$

$$\Rightarrow h = \frac{22}{9}$$

Thus, the value of h is $\frac{22}{9}$.

Question 11:

Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is $A(x - x_1) + B(y - y_1) = 0$

Answer 11:

The slope of line $Ax+By+C=0$ or $y=\left(\frac{-A}{B}\right)x+\left(\frac{-C}{B}\right)$ is $m=-\frac{A}{B}$

It is known that parallel lines have the same slope.

$\therefore$ Slope of the other line $=m=-\frac{A}{B}$

The equation of the line passing through point (x_1,y_1) and having a slope $m=-\frac{A}{B}$ is

$$y-y_1=m(x-x_1)$$

$$y-y_1=-\frac{A}{B}(x-x_1)$$

$$B(y-y_1)=-A(x-x_1)$$

$$A(x-x_1)+B(y-y_1)=0$$

Hence, the line through point (x_1,y_1) and parallel to line $Ax+By+C=0$ is A

$$(x-x_1)+B(y-y_1)=0$$

Question 12:

Two lines passing through the point $(2,3)$ intersects each other at an angle of 60°. If slope of one line is 2 , find equation of the other line.

Answer 12:

It is given that the slope of the first line, $m_1=2$.

Let the slope of the other line be m_2.

The angle between the two lines is 60°.

$$\therefore \tan 60^\circ=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\Rightarrow \sqrt{3}=\left|\frac{2-m_2}{1+2m_2}\right|$$

$$\Rightarrow \sqrt{3}=\pm\left(\frac{2-m_2}{1+2m_2}\right)$$

$$\Rightarrow \sqrt{3}=\frac{2-m_2}{1+2m_2} \text{ or } \sqrt{3}=-\left(\frac{2-m_2}{1+2m_2}\right)$$

$$\Rightarrow \sqrt{3}\left(1+2m_2\right)=2-m_2 \text{ or } \sqrt{3}\left(1+2m_2\right)=-\left(2-m_2\right)$$

$$\Rightarrow \sqrt{3}+2\sqrt{3}m_2+m_2=2 \text{ or } \sqrt{3}+2\sqrt{3}m_2-m_2=-2$$

$$\Rightarrow \sqrt{3}+(2\sqrt{3}+1)m_2=2 \text{ or } \sqrt{3}+(2\sqrt{3}-1)m_2=-2$$

$$\Rightarrow m_2=\frac{2-\sqrt{3}}{(2\sqrt{3}+1)} \text{ or } m_2=\frac{-(2+\sqrt{3})}{(2\sqrt{3}-1)}$$

Case I:

$$m_2=\left(\frac{2-\sqrt{3}}{2\sqrt{3}+1}\right)$$

The equation of the line passing through point $(2,3)$ and having a slope of $\frac{(2-\sqrt{3})}{(2\sqrt{3}+1)}$ is given by

$$(y-3)=\frac{2-\sqrt{3}}{2\sqrt{3}+1}(x-2)$$

$$(2\sqrt{3}+1)y-3(2\sqrt{3}+1)=(2-\sqrt{3})x-2(2-\sqrt{3})$$

$$(\sqrt{3}-2)x+(2\sqrt{3}+1)y=-4+2\sqrt{3}+6\sqrt{3}+3$$

$$(\sqrt{3}-2)x+(2\sqrt{3}+1)y=-1+8\sqrt{3}$$

In this case, the equation of the other line is $(\sqrt{3}-2)x+(2\sqrt{3}+1)y=-1+8\sqrt{3}$.

Case II : $m_2=\frac{-(2+\sqrt{3})}{(2\sqrt{3}-1)}$

The equation of the line passing through point $(2,3)$ and having a slope of $\frac{-(2+\sqrt{3})}{(2\sqrt{3}-1)}$ is

$$(y-3)=\frac{-(2+\sqrt{3})}{(2\sqrt{3}-1)}(x-2)$$

$$(2\sqrt{3}-1)y-3(2\sqrt{3}-1)=-(2+\sqrt{3})x+2(2+\sqrt{3})$$

$$(2\sqrt{3}-1)y+(2+\sqrt{3})x=4+2\sqrt{3}+6\sqrt{3}-3$$

$$(2+\sqrt{3})x+(2\sqrt{3}-1)y=1+8\sqrt{3}$$

In this case, the equation of the other line is $(2+\sqrt{3})x+(2\sqrt{3}-1)y=1+8\sqrt{3}$

Thus, the required equation of the other line is $(\sqrt{3}-2)x+(2\sqrt{3}+1)y=-1+8\sqrt{3}$ or

$(2+\sqrt{3})x+(2\sqrt{3}-1)y=1+8\sqrt{3}$

Question 13:

Find the equation of the right bisector of the line segment joining the points $(3,4)$ and $(-\ 1,2)$.

Answer 13:

The right bisector of a line segment bisects the line segment at 90°.

The end-points of the line segment are given as $A(3,4)$ and $B(-1,2)$.

Accordingly, mid-point of $AB=\left(\frac{3-1}{2},\frac{4+2}{2}\right)=(1,3)$

Slope of $AB=\frac{2-4}{-1-3}=\frac{-2}{-4}=\frac{1}{2}$

Slope of the line perpendicular to $AB=-\frac{1}{\left(\frac{1}{2}\right)}=-2$

The equation of the line passing through $(1,3)$ and having a slope of -2 is

$(y-3)=-2(x-1)y$

$-3=-2x+2$

$2x+y=5$

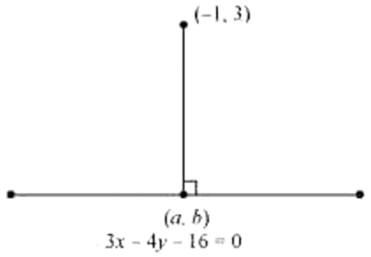
Thus, the required equation of the line is $2x+y=5$.

Question 14:

Find the coordinates of the foot of perpendicular from the point $(-1,3)$ to the line $3x-4y\ -16=0$

Answer 14:

Let (a,b) be the coordinates of the foot of the perpendicular from the point $(-1,3)$ to the line $3x-4y-16=0$.



Slope of the line joining $(-1,3)$ and $(a,b), m_1=\frac{b-3}{a+1}$

Slope of the line $3x-4y-16=0$ or $y=\frac{3}{4}x-4, m_2=\frac{3}{4}$

Since these two lines are perpendicular, $m_1 m_2=-1$

$$\therefore\left(\frac{b-3}{a+1}\right)\times\left(\frac{3}{4}\right)=-1$$

$$\Rightarrow\frac{3b-9}{4a+4}=-1$$

$$\Rightarrow 3b-9=-4a-4$$

$$\Rightarrow 4a+3b=5$$

Point (a,b) lies on line $3x-4y=16$

$$\therefore 3a-4b=16\ldots(2)$$

$$a=\frac{68}{25}\text{ and } b=-\frac{49}{25}$$

Thus, the required coordinates of the foot of the perpendicular are $\left(\frac{68}{25},-\frac{49}{25}\right)$.

Question 15:

The perpendicular from the origin to the line $y=mx+c$ meets it at the point $(-1,2)$. Find the values of m and c.

Answer 15:

The given equation of line is $y=mx+c$.

It is given that the perpendicular from the origin meets the given line at $(-1,2)$.

Therefore, the line joining the points $(0,0)$ and $(-1,2)$ is perpendicular to the given line.

$\therefore$ Slope of the line joining $(0,0)$ and $(-1,2)=\frac{2}{-1}=-2$

The slope of the given line is m.

$\therefore m\times -2=-1$

$\Rightarrow m=\frac{1}{2}$

Since point $(-1,2)$ lies on the given line, it satisfies the equation $y=mx+c$

$\therefore 2=m(-1)+c$

$\Rightarrow 2=\frac{1}{2}(-1)+c$

$\Rightarrow c=2+\frac{1}{2}=\frac{5}{2}$

Thus, the respective values of m and c are $\frac{1}{2}$ and $\frac{5}{2}$.

Question 16:

If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta-y\sin\theta$ $=k\cos 2\theta$ and $x\sec\theta+y\operatorname{cosec}\theta=k$, respectively, prove that $p^2+4q^2=k^2$

Answer 16:

The equations of given lines are

$x\cos\theta-y\sin\theta=k\cos 2\theta$

$x\sec\theta+y\operatorname{cosec}\theta=k$

The perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

$Ax+By+C=0$,

we obtain $A=\cos\theta, B=-\sin\theta$,

and $C=-k\cos 2\theta$

It is given that p is the length of the perpendicular from $(0,0)$ to line (1).

$$\therefore p=\frac{|A(0)+B(0)+C|}{\sqrt{A^2+B^2}}=\frac{|C|}{\sqrt{A^2+B^2}}=\frac{|-k\cos 2\theta|}{\sqrt{\cos^2\theta+\sin^2\theta}}=|-k\cos 2\theta|$$

$Ax+By+C=0$, we obtain $A=\sec\theta, B=\operatorname{cosec}\theta$, and $C=-k$

It is given that q is the length of the perpendicular from $(0,0)$ to line (2).

$$\therefore q=\frac{|A(0)+B(0)+C|}{\sqrt{A^2+B^2}}=\frac{|C|}{\sqrt{A^2+B^2}}=\frac{|-k|}{\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}}$$

We have,

$$p^2+4q^2=(|-k\cos 2\theta|)^2+4\left(\frac{|-k|}{\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}}\right)^2$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\sec^2\theta+\operatorname{cosec}^2\theta\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{1}{\sin^2\theta\cos^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+4k^2\sin^2\theta\cos^2\theta$$

$$=k^2\cos^2 2\theta+k^2(2\sin\theta\cos\theta)^2$$

$$=k^2\cos^2 2\theta+k^2\sin^2 2\theta$$

$$=k^2\left(\cos^2 2\theta+\sin^2 2\theta\right)$$

$$=k^2$$

Hence, we proved that $p^2+4q^2=k^2$.

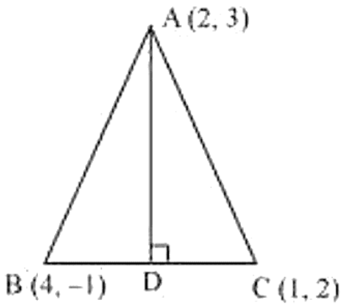
Question 17:

In the triangle A B C with vertices $A(2,3), B(4,-1)$ and $C(1,2)$, find the equation and length of altitude from the vertex A.

Answer 17:

Let A D be the altitude of triangle A B C from vertex A.

Accordingly, AD perpendicular to BC



The equation of the line passing through point $(2,3)$ and having a slope of 1 is

$(y-3)=1(x-2)$

$\Rightarrow x-y+1=0$

$\Rightarrow y-x=1$

Therefore, equation of the altitude from vertex $A=y-x=1$.

Length of $AD=$ Length of the perpendicular from $A(2,3)$ to B C The equation of B C is

$(y+1)=\frac{2+1}{1-4}(x-4)$

$\Rightarrow (y+1)=-1(x-4)$

$\Rightarrow y+1=-x+4$

$\Rightarrow x+y-3=0$

The perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by

$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

$Ax+By+C=0$, we obtain $A=1, B=1$, and $C=-3$

$\therefore$ Length of $AD=\frac{|1\times 2+|\times 3-3|}{\sqrt{1^2+1^2}}$ units $=\frac{|2|}{\sqrt{2}}$ units $=\frac{2}{\sqrt{2}}$ units $=\sqrt{2}$ units

Thus, the equation and the length of the altitude from vertex A are $y-x=1$ and $\sqrt{2}$ units respectively.

Question 18:

If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that: $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$

Answer 18:

It is known that the equation of a line whose intercepts on the axes are a and b is

$$\frac{x}{a}+\frac{y}{b}=1$$

or $bx+ay=ab$

or $bx+ay-ab=0$

The perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

$Ax+By+C=0$, we obtain $A=b, B=a$, and $C=-ab$

Therefore, if p is the length of the perpendicular from point $(x_1,y_1)=(0,0)$ to line (1), we obtain

$$p=\frac{|A(0)+B(0)-ab|}{\sqrt{b^2+a^2}}$$

$$\Rightarrow p=\frac{|-ab|}{\sqrt{a^2+b^2}}$$

On squaring both sides, we obtain

$$p^2=\frac{(-ab)^2}{a^2+b^2}$$

$$\Rightarrow p^2\left(a^2+b^2\right)=a^2b^2$$

$$\Rightarrow \frac{a^2+b^2}{a^2b^2}=\frac{1}{p^2}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

Hence, we showed that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Miscellaneous Examples

Example 20

If the lines $2x+y-3=0, 5x+ky-3=0$ and $3x-y-2=0$ are concurrent, find the value of k.

Solution

Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$2x+y-3=0$

$5x+ky-3=0$

$3x-y-2=0$

Isolate x for $2x+y-3=5$: $x=\frac{-y+8}{2}$

Substitute $x=\frac{-y+8}{2}$

$\frac{-y+8}{2}-y-2=0$

$\frac{-y+8}{2}+ky-3=3$

Simplify

$\frac{-3y+8}{2}-2=0$

$\frac{-y+8}{2}+ky-3=3$

Isolate y for $\frac{-3y+8}{2}-2=0$: $y=\frac{4}{3}$

Substitute $y=\frac{4}{3}$

$$\left[\frac{-\frac{4}{3}+8}{2}+k\frac{4}{3}-3=3\right]$$

Simplify

$\left[\frac{1}{3}+\frac{4}{3}k=3\right]$

Isolate k for $\frac{1}{3}+\frac{4}{3}k=3$: $\quad k=2$

$x=\frac{-\frac{4}{3}+8}{2}$

$\frac{-\frac{4}{3}+8}{2}=\frac{10}{3}$

$x=\frac{10}{3}$

The solutions to the system of equations are: $x=\frac{10}{3}, k=2, y=\frac{4}{3}$

$k=2$

Example 21

Find the distance of the line $4x-y=0$ from the point $\mathrm{P}(4,1)$ measured along the line making an angle of 135° with the positive x-axis.

Solution

Given line is $4x-y=0$ In order to find the distance of the line (1) from the point $\mathrm{P}(4,1)$ along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 10.21). Slope of second line is $\tan 135^{\circ}=-1$. Equation of the line with slope -1 through the point $\mathrm{P}(4,1)$ is

$y-1=-1(x-4)$ or $x+y-5=0$

Solving this we get

$x=1$ and $y=4$ so that point of intersection of the two lines is $\mathrm{Q}(1,4)$.

Now, distance of line (1) from the point $\mathrm{P}(4,1)$ along the line (2)

$=$ the distance between the points $\mathrm{P}(4,1)$ and $\mathrm{Q}(1,4)$

$\sqrt{(1-4)^2+(4-1)^2}$

$(1-4)^2=3^2$

$(4-1)^2 = 3^2$

$= \sqrt{3^2 + 3^2}$

Add similar elements: $3^2 + 3^2 = 3^2 \cdot 2$

$= \sqrt{3^2 \cdot 2}$

Apply radical rule: $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b},$ assuming $a \geq 0, b \geq 0$ $= \sqrt{2}\sqrt{3^2}$

Apply radical rule: $\sqrt[n]{a^n} = a,$ assuming $a \geq 0$

$\sqrt{3^2} = 3$

$= 3\sqrt{2}$

Example 22

Assuming that straight lines work as the plane mirror for a point, find the image of the point $(1, 2)$ in the line $x - 3y + 4 = 0$

Solution

Let $\mathrm{Q}(h, k)$ is the image of the point $\mathrm{P}(1, 2)$ in the line

$x - 3y + 4 = 0$

Hence Slope of line $\mathrm{PQ} = \dfrac{-1}{\text{Slope of line } x - 3y + 4 = 0}$,

so that $\dfrac{k-2}{h-1} = \dfrac{-1}{\frac{1}{3}}$ or $3h + k = 5$

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0$$

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0$$

Remove parentheses: $(a) = a$

$$\frac{h+1}{2} - 3 \cdot \frac{k+2}{2} + 4 = 0$$

Multiply $3 \cdot \frac{k+2}{2} : \frac{3(k+2)}{2}$

$$\frac{h+1}{2} - \frac{3(k+2)}{2} + 4 = 0$$

Add $\frac{3(k+2)}{2}$ to both sides

$$\frac{h+1}{2} - \frac{3(k+2)}{2} + 4 + \frac{3(k+2)}{2} = 0 + \frac{3(k+2)}{2}$$

Simplify

$$\frac{h+1}{2} + 4 = \frac{3(k+2)}{2}$$

Subtract 4 from both sides

$$\frac{h+1}{2} + 4 - 4 = \frac{3(k+2)}{2} - 4$$

Simplify

$$\frac{h+1}{2} = \frac{3(k+2)}{2} - 4$$

$$h - 3k = -3$$

Solving (2) and (3), we get $h = \frac{6}{5}$ and $k = \frac{7}{5}$.

Hence, the image of the point $(1, 2)$ in the line (1) is $\left(\frac{6}{5}, \frac{7}{5}\right)$.

Example 23

Show that the area of the triangle formed by the lines

$y = m_1 x + c_1, y = m_2 x + c_2$ and $x = 0$ is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$

Solution

Given lines are

$$y = m_1 x + c_1$$

$$y = m_2 x + c_2$$

$$x = 0$$

We know that line $y = mx + c$ meets the line $x = 0$ (y -axis) at the point $(0, c)$.

Therefore, two vertices of the triangle formed by lines (1) to (3) are $\mathrm{P}(0, c_1)$ and $\mathrm{Q}(0, c_2)$ (Fig 10. 23). Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get

$$x = \frac{(c_2 - c_1)}{(m_1 - m_2)} \text{ and } y = \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}$$

Therefore, third vertex of the triangle is $\mathrm{R}\left(\frac{(c_2 - c_1)}{(m_1 - m_2)}, \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}\right)$.

Now, the area of the triangle is

$$\$ = \frac{1}{2} | \; 0\left(\frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} - c_2\right) + \frac{c_2 - c_1}{m_1 - m_2}(c_2 - c_1) + 0\left(c_1 - \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right) = \frac{(c_2 - c_1)^2}{2|m_1 - m_2|} \$$$

Example 24

A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.

Solution

Given lines are

$$5x - y + 4 = 0$$

$$3x + 4y - 4 = 0$$

Let the required line intersects the lines (1) and (2) at the points, (α_1, β_1) and (α_2, β_2)

Therefore

$5\alpha_1 - \beta_1 + 4 = 0$ and

$$3\alpha_2 + 4\beta_2 - 4 = 0$$

or $\beta_1 = 5\alpha_1 + 4$ and $\beta_2 = \frac{4 - 3\alpha_2}{4}$.

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is $(1, 5)$. Therefore

$$\frac{\alpha_1 + \alpha_2}{2} = 1 \text{ and } \frac{\beta_1 + \beta_2}{2} = 5$$

or $\alpha_1+\alpha_2=2$ and $\dfrac{5\alpha_1+4+\dfrac{4-3\alpha_2}{4}}{2}=5,$

or $\alpha_1+\alpha_2=2$ and $20\alpha_1-3\alpha_2=20$

Solving equations in (3) for α_1 and α_2, we get

$\alpha_1=\dfrac{26}{23}$ and $\alpha_2=\dfrac{20}{23}$ and

hence, $\beta_1=5\cdot\dfrac{26}{23}+4=\dfrac{222}{23}$.

Equation of the required line passing through $(1,5)$ and (α_1,β_1) is

$y-5=\dfrac{\beta_1-5}{\alpha_1-1}(x-1)$ or $y-5=\dfrac{\dfrac{222}{23}-5}{\dfrac{26}{23}-1}(x-1)$

or

$107x-3y-92=0$

which is the equation of required line.

Example 25

Show that the path of a moving point such that its distances from two lines $3x-2y=5$ and $3x+2y=5$ are equal is a straight line.

Solution

Given lines are

$3x-2y=5$

and

$3x+2y=5$

Let (h,k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$\dfrac{|3h-2k-5|}{\sqrt{9+4}}=\dfrac{|3h+2k-5|}{\sqrt{9+4}}$ or $|3h-2k-5|=|3h+2k-5|$

which gives $3h-2k-5=3h+2k-5$ or $-(3h-2k-5)=3h+2k-5$.

Solving these two relations we get $k=0$ or $h=\dfrac{5}{3}$.

Thus, the point (h, k) satisfies the equations $y=0$ or $x=\frac{5}{3}$,

which represent straight lines.

Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

Miscellaneous Exercise on Chapter 10

Question 1:

Find the values of k for which the line $(k-3)x-\left(4-k^2\right)y+k^2-7k+6=0$ is

(a) Parallel to the x-axis,

(b) Parallel to the y-axis,

(c) Passing through the origin.

Answer 1:

The given equation of line is

$$(k-3)x-\left(4-k^2\right)y+k^2-7k+6=0\ldots(1)$$

(a) If the given lines are parallel to x axis, then slope of the given line =Slope of the x-axis

The given line can be written as

$$\left(4-k^2\right)y=(k-3)x+k^2-7k+6=0$$

$$y=\frac{(k-3)}{\left(4-k^2\right)}x+\frac{k^2-7k+6}{\left(4-k^2\right)}$$, which is of the form $y=mx+c$

Slope of the given line $=\frac{(k-3)}{\left(4-k^2\right)}$

Slope of the x-axis $=0$

$$\therefore \frac{(k-3)}{\left(4-k^2\right)}=0$$

$$\Rightarrow k-3=0$$

$$\Rightarrow k=3$$

Thus, if the given line is parallel to the x-axis, then the value of k is $3 .

(b) If the given lines are parallel to the y axis , its slope will be undefined.

Then

$$\frac{(k-3)}{\left(4-k^2\right)}.$$

Now, $\frac{(k-3)}{\left(4-k^2\right)}$ is undefined at $k^2=4$

$$k^2=4$$

$$\Rightarrow k=\pm 2$$

Thus, if the given line is parallel to the y-axis, then the value of k is ± 2.

(c) If the given line is passing through the origin, then point $(0,0)$ satisfies the given equation of line.

$$(k-3)(0)-\left(4-k^2\right)(0)+k^2-7k+6=0$$

$$k^2-7k+6=0$$

$$k^2-6k-k+6=0$$

$$(k-6)(k-1)=0$$

$$k=1 \text{ or } 6$$

Thus, if the given line is passing through the origin, then the value of k is either 1 or 6 .

Question 2:

Find the values of θ and p, if the equation $x\cos\theta+y\sin\theta=p$ is the normal form of the line $\sqrt{3}x+y+2=0$

Answer 2:

The equation of the given line is $\sqrt{3}x+y+2=0$.

This equation can be reduced as

$$\sqrt{3}x+y+2=0$$

$$\Rightarrow -\sqrt{3}x-y=2$$

On dividing both sides by $\sqrt{(-\sqrt{3})^2+(-1)^2}=2$, we obtain

$$-\frac{\sqrt{3}}{2}x-\frac{1}{2}y=\frac{2}{2}$$

$$\Rightarrow\left(-\frac{\sqrt{3}}{2}\right)x+\left(-\frac{1}{2}\right)y=1$$

Comparing equation (1) to $x\cos\theta+y\sin\theta=p$, we obtain

$$\cos\theta=-\frac{\sqrt{3}}{2},\sin\theta=-\frac{1}{2}, \text{ and } p=1$$

Since the values of $\sin\theta$ and $\cos\theta$ are negative,

$$\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$$

Thus, the respective values of θ and p are $\frac{7\pi}{6}$ and 1 .

Question 3:

Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6, respectively.

Answer 3:

Let the intercepts cut by the given lines on the axes be a and b.

It is given that

$a+b=1$.................(1)

$ab=-6$...................(2)

On solving equations (1) and (2), we obtain

$a=3$ and $b=-2$ or $a=-2$ and $b=3$

It is known that the equation of the line whose intercepts on the axes are a and b is $\frac{x}{a}+\frac{y}{b}=1$ or

$bx+ay-ab=0$

Case I: $a=3$ and $b=-2$

In this case, the equation of the line is $-2x+3y+6=0$, i.e., $2x-3y=6$

Case II: $a=-2$ and $b=3$

In this case, the equation of the line is $3x-2y+6=0$, i.e., $-3x+2y=6$

Thus, the required equation of the lines are $2x-3y=6$ and $-3x+2y=6$

Question 4:

What are the points on the y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

Answer 4:

Let $(0,b)$ be the point on the y-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units. The given line can be written as

$4x+3y-12=0$

$Ax+By+C=0$, we obtain $A=4, B=3$, and $C=-12$

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

Therefore, if $(0,b)$ is the point on the y-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units, then:

$$4=\frac{|4(0)+3(b)-12|}{\sqrt{4^2+3^2}}$$

$$\Rightarrow 4=\frac{|3b-12|}{5}$$

$$\Rightarrow 20=|3b-12|$$

$$\Rightarrow 20=\pm(3b-12)$$

$$\Rightarrow 20=(3b-12) \text{ or } 20=-(3b-12)$$

$$\Rightarrow 3b=20+12 \text{ or } 3b=-20+12$$

$$\Rightarrow b=\frac{32}{3} \text{ or } b=-\frac{8}{3}$$

Thus, the required points are $\left(0,\frac{32}{3}\right)$ and $\left(0,-\frac{8}{3}\right)$

Question 5:

Find the perpendicular distance from the origin to the line joining the points $(\cos\theta,\sin\theta)$ and $(\cos\phi,\sin\phi)$

Answer 5:

The equation of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is given by

$$y-\sin\theta=\frac{\sin\phi-\sin\theta}{\cos\phi-\cos\theta}(x-\cos\theta)$$

$$y(\cos\phi-\cos\theta)-\sin\theta(\cos\phi-\cos\theta)=x(\sin\phi-\sin\theta)-\cos\theta(\sin\phi-\sin\theta)$$

$$x(\sin\theta-\sin\phi)+y(\cos\phi-\cos\theta)+\cos\theta\sin\phi-\cos\theta\sin\theta-\sin\theta\cos\phi+\sin\theta\cos\theta=0$$

$$x(\sin\theta-\sin\phi)+y(\cos\phi-\cos\theta)+\sin(\phi-\theta)=0$$

$$Ax+By+C=0,$$

where $A=\sin\theta-\sin\phi, B=\cos\phi-\cos\theta$, and $C=\sin(\phi-\theta)$

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1, y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

Therefore, the perpendicular distance (d) of the given line from point $(x_1, y_1)=(0,0)$ is

$$d=\frac{|(\sin\theta-\sin\phi)(0)+(\cos\phi-\cos\theta)(0)+\sin(\phi-\theta)|}{\sqrt{(\sin\theta-\sin\phi)^2+(\cos\phi-\cos\theta)^2}}$$

$$=\frac{|\sin(\phi-\theta)|}{\sqrt{\sin^2\theta+\sin^2\phi-2\sin\theta\sin\phi+\cos^2\phi+\cos^2\theta-2\cos\phi\cos\theta}}=\frac{|\sin(\phi-\theta)|}{\sqrt{\left(\sin^2\theta+\cos^2\theta\right)+\left(\sin^2\phi+\cos^2\phi\right)-2(\sin\theta}}$$

$$=\frac{|\sin(\phi-\theta)|}{\sqrt{1+1-2(\cos(\phi-\theta))}}$$

$$=\frac{|\sin(\phi-\theta)|}{\sqrt{2(1-\cos(\phi-\theta))}}$$

$$=\frac{|\sin(\phi-\theta)|}{\sqrt{2\left(2\sin^2\left(\frac{\phi-\theta}{2}\right)\right)}}$$

$$=\frac{|\sin(\phi-\theta)|}{2\sin(\phi-\theta)|}$$

Question 6:

Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x-7y+5=0$ and $3x+y=0$

Answer 6:

The equation of any line parallel to the y-axis is of the form

$x=a$

The two given lines are

$x-7y+5=0$

$3x+y=0$

$3x+y=0$ On solving equations (2) and (3), we obtain $x=-\frac{5}{22}$ and $y=\frac{15}{22}$

Therefore, $\left(-\frac{5}{22},\frac{15}{22}\right)$ is the point of intersection of lines (2) and (3).

Since line $x=$ a passes through point $\left(-\frac{5}{22},\frac{15}{22}\right)$, So, $a=-\frac{5}{22}$

Thus, the required equation of the line is $x=-\frac{5}{22}$

Question 7:

Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis.

Answer 7:

The equation of the given line is $\frac{x}{4}+\frac{y}{6}=1$.

This equation can also be written as $3x+2y-12=0$

$y=\frac{-3}{2}x+6$, which is of the form $y=mx+c$

$\therefore$ Slope of the $=-\frac{3}{2}$ given line

$\therefore$ Slope of line perpendicular to the given line $=-\frac{1}{\left(-\frac{3}{2}\right)}=\frac{2}{3}$

Let the given line intersect the y-axis at $(0, y)$.

On substituting x with 0 in the equation of the given line, we obtain $\frac{y}{6} = 1 \Rightarrow y = 6$

$\therefore$ The given line intersects the y-axis at $(0,6)$.

The equation of the line that has a slope of $2/3$ and passes through point $(0,6)$ is

$$(y-6) = \frac{2}{3}(x-0)$$

$$3y - 18 = 2x$$

$$2x - 3y + 18 = 0$$

Thus, the required equation of the line is $2x - 3y + 18 = 0$.

Question 8:

Find the area of the triangle formed by the lines $y - x = 0, x + y = 0$ and $x - k = 0$.

Answer 8:

The equations of the given lines are

$$y - x = 0$$

$$x + y = 0$$

$$x - k = 0$$

The point of intersection of lines (1) and (2) is given by $x = 0$ and $y = 0$

The point of intersection of lines (2) and (3) is given by $x = k$ and $y = -k$

The point of intersection of lines (3) and (1) is given by $x = k$ and $y = k$

Thus,

the vertices of the triangle formed by the three given lines are $(0,0), (k,-k)$, and (k,k)

We know that the area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) is

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Therefore,

area of the triangle formed by the three given lines

$$= \frac{1}{2}|\,0(-k-k) + k(k-0) + k(0+k)\,| \text{ square units}$$

$=\frac{1}{2}\left|k^2+k^2\right|$ square units

$=\frac{1}{2}\left|2k^2\right|$ square units

$=k^2$ square units

Question 9:

Find the value of p so that the three lines $3x+y-2=0, px+2y-3=0$ and $2x-y-3\ =0$ may intersect at one point.

Answer 9:

The equations of the given lines are

$3x+y-2=0$

$px+2y-3=0$

$2x-y-3=0$

On solving equations (1) and (3), we obtain

$x=1$ and $y=-1$

$p(1)+2(-1)-3=0$

$p-2-3=0 p=5$

Thus, the required value of p is 5 .

Question 10:

If three lines whose equations are $y=m_1x+c_1, y=m_2x+c_2$ and $y=m_3x+c_3$ are concurrent, then show that $m_1\left(c_2-c_3\right)+m_2\left(c_3-c_1\right)+m_3\left(c_1-c_2\right)=0$.

Answer 10:

The equations of the given lines are

$y=m_1x+c_1$ …………………(1)

$y=m_2x+c_2$

$y=m_3x+c_3$

On subtracting equation (1) from (2), we obtain

$0=\left(m_2-m_1\right)x+\left(c_2-c_1\right)$

$$\Rightarrow (m_1 - m_2)x = c_2 - c_1$$

$$\Rightarrow x = \frac{c_2 - c_1}{m_1 - m_2}$$

On substituting this value of x in (1), we obtain

$$y = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$

$$y = \frac{m_1c_2 - m_1c_1}{m_1 - m_2} + c_1$$

$$y = \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{m_1 - m_2}$$

$$y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

$\therefore \left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$ is the point of intersection of lines (1) and (2).

It is given that lines $(1), (2)$, and (3) are concurrent. Hence, the point of intersection of lines (1) and (2) will also satisfy equation (3).

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_3$$

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + c_3m_1 - c_3m_2}{m_1 - m_2}$$

$$m_1c_2 - m_2c_1 - m_3c_2 + m_3c_1 - c_3m_1 + c_3m_2 = 0$$

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

Question 11:

Find the equation of the lines through the point $(3, 2)$ which make an angle of 45° with the line $x - 2y = 3$

Answer 11:

Let the slope of the required line be m_1.

The given line can be written as $y = \frac{1}{2}x - \frac{3}{2}$, which is of the form $y = mx + c$

$\therefore$ Slope of the given line $= m_2 = \frac{1}{2}$

It is given that the angle between the required line and line $x-2y=3$ is 45°.

We know that if θ is the acute angle between lines I_1 and I_2 with slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right|$$

$$\therefore \tan 45^\circ = \frac{|m_1 - m_2|}{1+m_1m_2}$$

$$\Rightarrow 1 = \left|\frac{1}{2} - m_1\right|$$

$$1+\frac{m_1}{2}|$$

$$\Rightarrow 1 = |\left(\frac{1-2m_1}{2}\right)|$$

$$\Rightarrow 1 = \frac{2+m_1}{2}|$$

$$\Rightarrow \left|\frac{1-2m_1}{2+m_1}\right|$$

$$\Rightarrow 1 = \pm\left(\frac{1-2m_1}{2+m_1}\right)$$

$$\Rightarrow \frac{1-2m_1}{2+m_1} \text{ or } 1 = -\left(\frac{1-2m_1}{2+m_1}\right)$$

$$\Rightarrow 1-2m_1 \text{ or } 2+m_1 = -1+2m_1$$

$$\Rightarrow \frac{1}{3} \text{ or } m_1 = 3$$

Case I: $m_1 = 3$

The equation of the line passing through $(3,2)$ and having a slope of 3 is:

$$y-2 = 3(x-3)$$

$$y-2 = 3x-9$$

$$3x - y = 7$$

Case II: $m_1 = -\frac{1}{3}$

The equation of the line passing through $(3, 2)$ and having a slope of $-\frac{1}{3}$ is:

$$y - 2 = -\frac{1}{3}(x - 3)$$

$$3y - 6 = -x + 3$$

$$x + 3y = 9$$

Thus, the equations of the lines are $3x - y = 7$ and $x + 3y = 9$.

Question 12:

Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

Answer 12:

Let the equation of the line having equal intercepts on the axes be

$$\frac{x}{a} + \frac{y}{a} = 1$$

Or $x + y = a$

On solving equations $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$, we obtain $x = \frac{1}{13}$ and $y = \frac{5}{13}$

$\therefore \left(\frac{1}{13}, \frac{5}{13}\right)$ is the point of intersection of the two given lines.

Since equation (1) passes through $\left(\frac{1}{13}, \frac{5}{13}\right)$ point, $\frac{1}{13} + \frac{5}{13} = a \Rightarrow a = \frac{6}{13}$

$\therefore$ Equation (1) becomes $x + y = \frac{6}{13}$, i.e., $13x + 13y = 6$

Thus, the required equation of the line is $13x + 13y = 6$

Question 13:

Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$

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Answer 13:

Let the equation of the line passing through the origin be $y = m_1 x$.

If this line makes an angle of θ with line $y = mx + c$, then angle θ is given by

$$\therefore \tan\theta = \left|\frac{m_1 - m}{1 + m_1 m}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}\right|$$

$$\Rightarrow \tan\theta = \pm\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}\right)$$

$$\Rightarrow \tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x} m} \text{ or } \tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{m}}\right)$$

Case I:

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}$$

$$\Rightarrow \tan\theta + \frac{y}{x} m \tan\theta = \frac{y}{x} - m$$

$$\Rightarrow m + \tan\theta = \frac{y}{x}(1 - m\tan\theta)$$

$$\Rightarrow \frac{y}{x} = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Case II:

$$\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}\right)$$

$$\Rightarrow \tan\theta + \frac{y}{x} m \tan\theta = -\frac{y}{x} + m$$

$$\Rightarrow \frac{y}{x}(1+m\tan\theta)=m-\tan\theta$$

$$\Rightarrow \frac{y}{x}=\frac{m-\tan\theta}{1+m\tan\theta}$$

Therefore, the required line is given by $\frac{y}{x}=\frac{m\pm\tan\theta}{1\mp m\tan\theta}$

Question 14:

In what ratio, the line joining $(-1,1)$ and $(5,7)$ is divided by the line $x+y=4$?

Answer 14:

The equation of the line joining the points $(-1,1)$ and $(5,7)$ is given by

$$y-1=\frac{7-1}{5+1}(x+1)$$

$$y-1=\frac{6}{6}(x+1)$$

$$x-y+2=0$$

The equation of the given line is

$$x+y-4=0$$

The point of intersection of lines (1) and (2) is given by

$$x=1 \text{ and } y=3$$

Let point $(1,3)$ divide the line segment joining $(-1,1)$ and $(5,7)$ in the ratio 1: k.

Accordingly, by section formula,

$$(1,3)=\left(\frac{k(-1)+1(5)}{1+k},\frac{k(1)+1(7)}{1+k}\right)$$

$$\Rightarrow (1,3)=\left(\frac{-k+5}{1+k},\frac{k+7}{1+k}\right)$$

$$\Rightarrow \frac{-k+5}{1+k}=1, \frac{k+7}{1+k}=3$$

$$\therefore \frac{-k+5}{1+k}=1$$

$$\Rightarrow -k+5=1+k$$

$\Rightarrow 2k = 4$

$\Rightarrow k = 2$

Thus, the line joining the points $(-1,1)$ and $(5,7)$ is divided by line $x + y = 4$ in the ratio 1: 2.

Question 15:

Find the distance of the line $4x+7y+5=0$ from the point $(1,2)$ along the line $2x-y=0$.

Answer 15:

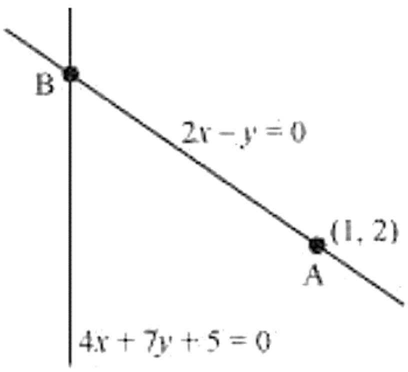
The given lines are

$2x - y = 0$

$4x + 7y + 5 = 0$

$A(1,2)$ is a point on line (1).

Let B be the point of intersection of lines (1) and (2).



On solving equations (1) and (2), we obtain $x = \frac{-5}{18}$ and $y = \frac{-5}{9}$ $\therefore$ Coordinates of point B are $\left(\frac{-5}{18}, \frac{-5}{9}\right)$

By using distance formula, the distance between points A and B can be obtained as

$$AB = \sqrt{\left(1+\frac{5}{18}\right)^2 + \left(2+\frac{5}{9}\right)^2} \text{ units}$$

$$= \sqrt{\left(\frac{23}{18}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

$$=\sqrt{\left(\frac{23}{2\times 9}\right)^2+\left(\frac{23}{9}\right)^2}\text{ units}$$

$$=\sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{2}\right)^2+\left(\frac{23}{9}\right)^2}\text{ units}$$

$$=\sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{4}+1\right)}\text{ units}$$

$$=\frac{23}{9}\sqrt{\frac{5}{4}}\text{ units}$$

$$=\frac{23}{9}\times\frac{\sqrt{5}}{2}\text{ units}$$

$$=\frac{23\sqrt{5}}{18}\text{ units}$$

Thus, the required distance is $\frac{23\sqrt{5}}{18}$ units

Question 16:

Find the direction in which a straight line must be drawn through the point $(-1,2)$ so that its point of intersection with the line $x+y=4$ may be at a distance of 3 units from this point.

Answer 16:

Let $y=mx+c$ be the line through point $(-1,2)$.

Accordingly, $2=m(-1)+c$

$\Rightarrow 2=-m+c$

$\Rightarrow c=m+2$

$\therefore y=mx+m+2$

The given line is

$x+y=4$

On solving the equations, we obtain

$x=\frac{2-m}{m+1}$ and $y=\frac{5m+2}{m+1}$

$\therefore \left(\frac{2-m}{m+1}, \frac{5m+2}{m+1}\right)$ is the point of intersection of lines (1) and (2)

Since this point is at a distance of 3 units from point $(-1,2)$, according to distance formula,

$$\sqrt{\left(\frac{2-m}{m+1}+1\right)^2+\left(\frac{5m+2}{m+1}-2\right)^2}=3$$

$$\Rightarrow \left(\frac{2-m+m+1}{m+1}\right)^2+\left(\frac{5m+2-2m-2}{m+1}\right)^2=3^2$$

$$\Rightarrow \frac{9}{(m+1)^2}+\frac{9m^2}{(m+1)^2}=9$$

$$\Rightarrow \frac{1+m^2}{(m+1)^2}=1$$

$$\Rightarrow 1+m^2=m^2+1+2m$$

$$\Rightarrow 2m=0$$

$$\Rightarrow m=0$$

Thus, the slope of the required line must be zero i.e., the line must be parallel to the x axis.

Question 17

The hypotenuse of a right angled triangle has its ends at the points $(1,3)$ and $(-4,1)$. Find the equation of the legs (perpendicular sides) of the triangle.

Answer 17

Let ABC be the right angles triangle, where $\angle C=90^\circ$

There are infinity many such lines.

Let m be the slope of AC.

$\therefore$ Slope of $BC=\frac{1}{m}$

Equation of AC : $y-3=m(x-1)$

$$\Rightarrow \mathrm{x}-1=\frac{1}{\mathrm{m}}(\mathrm{y}-3)$$

Equation of $\mathrm{BC}: \mathrm{y}-1=-\frac{1}{\mathrm{m}}(\mathrm{x}+4)$

$$\Rightarrow \mathrm{x}+4=-\mathrm{m}(\mathrm{y}-1)$$

For a given value of m, we can get these equations

For $\mathrm{m}=0, \mathrm{y}-3=0; \mathrm{x}+4=0$

For $\mathrm{m} \to \infty, \mathrm{x}-1=0; \mathrm{y}-1=0$

Question 18:

Find the image of the point $(3,8)$ with respect to the line $x+3y=7$ assuming the line to be a plane mirror.

Answer 18:

The equation of the given line is $x \ +3y=7 \ldots$ (1)

Let point $B(a,b)$ be the image of point $A(3,8)$.

Accordingly, line (1) is the perpendicular bisector of A B

Slope of $\mathrm{AB}=\frac{b-8}{a-3}$,

while the slope of line $(1)=-\frac{1}{3}$

Since line (1) is perpendicular to A B,

$$\left(\frac{b-8}{a-3}\right)\times\left(-\frac{1}{3}\right)=-1$$

$$\Rightarrow \frac{b-8}{3a-9}=1$$

$$\Rightarrow b-8=3a-9$$

$$\Rightarrow 3a-b=1$$

Mid-point of $\mathrm{AB}=\left(\frac{a+3}{2},\frac{b+8}{2}\right)$

The mid-point of line segment A B also satisfy the line (1)

$$\left(\frac{a+3}{2}\right)+3\left(\frac{b+8}{2}\right)=7$$

$$\Rightarrow a+3+3b+24=14$$

$$\Rightarrow a+3b=-13$$

On solving equations (2) and (3), we obtain $a=-1$ and $b=-4$.

Thus, the image of the given point with respect to the given line is $(-1,-4)$.

Question 19:

If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

Answer 19:

The equations of the given lines are

$y = 3x + 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

$2y = x + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

$y = mx + 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Slope of line $(1), m_1 = 3$

Slope of line $(2), m_2 = 1/2$

Slope of line (3), $m_3 = m$

It is given that lines (1) and (2) are equally inclined to line (3). This means that the angle between lines (1) and (3) equals the angle between lines (2) and (3).

$$\therefore \left|\frac{m_1 - m_3}{1 + m_1 m_3}\right| = \left|\frac{m_2 - m_3}{1 + m_2 m_3}\right|$$

$$\Rightarrow \left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{1}{2} - m\right|1 + \frac{1}{2}|$$

$$\Rightarrow \left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{1 - 2m}{m + 2}\right|$$

$$\Rightarrow \frac{3 - m}{1 + 3m} = \pm\left(\frac{1 - 2m}{m + 2}\right)$$

$$\Rightarrow \frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2} \text{ or } \frac{3 - m}{1 + 3m} = -\left(\frac{1 - 2m}{m + 2}\right)$$

If $\frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2}$, then

$(3 - m)(m + 2) = (1 - 2m)(1 + 3m)$

$\Rightarrow -m^2 + m + 6 = 1 + m - 6m^2$

$\Rightarrow 5m^2 + 5 = 0$

$\Rightarrow (m^2 + 1) = 0$

$\Rightarrow m=\sqrt{-1}$, which is not real

Hence, this case is not posible.

If $\frac{3-m}{1+3m}=-\left(\frac{1-2m}{m+2}\right)$, then

$\Rightarrow (3-m)(m+2)=-(1-2m)(1+3m)$

$\Rightarrow -m^2+m+6=-\left(1+m-6m^2\right)$

$\Rightarrow 7m^2-2m-7=0$

$\Rightarrow m=\frac{2\pm\sqrt{4-4(7)(-7)}}{2(7)}$

$\Rightarrow m=\frac{2\pm 2\sqrt{1+49}}{14}$

$\Rightarrow m=\frac{1\pm 5\sqrt{2}}{7}$

Thus, the required value of m is $\frac{1\pm 5\sqrt{2}}{7}$

Question 20:

If sum of the perpendicular distances of a variable point $P(x, y)$ from the lines $x+y-5=0$ and $3x-2y+7=0$ is always 10. Show that P must move on a line.

Answer 20:

The equations of the given lines are x

$+y-5=0\ldots$ (1)

$3x-2y+7=0\ldots$ (2)

The perpendicular distances of $P(x, y)$ from lines (1) and (2) are respectively given by

$d_1=\frac{|x+y-5|}{\sqrt{(1)^2+(1)^2}}$ and $d_2=\frac{|3x-2y+7|}{\sqrt{(3)^2+(-2)^2}}$

i.e., $d_1=\frac{|x+y-5|}{\sqrt{2}}$ and $d_2=\frac{|3x-2y+7|}{\sqrt{13}}$

It is given that $d_1+d_2=10$

$$\therefore \frac{|x+y-5|}{\sqrt{2}}+\frac{|3x-2y+7|}{\sqrt{13}}=10$$

$$\Rightarrow \sqrt{13}|x+y-5|+\sqrt{2}|3x-2y+7|-10\sqrt{26}=0$$

$$\Rightarrow \sqrt{13}(x+y-5)+\sqrt{2}(3x-2y+7)-10\sqrt{26}=0$$

Assuming $(x+y-5)$ and $(3x-2y+7)$ are positive

$$\Rightarrow \sqrt{13}x+\sqrt{13}y-5\sqrt{13}+3\sqrt{2}x-2\sqrt{2}y+7\sqrt{2}-10\sqrt{26}=0$$

$$\Rightarrow x(\sqrt{13}+3\sqrt{2})+y(\sqrt{13}-2\sqrt{2})+(7\sqrt{2}-5\sqrt{13}-10\sqrt{26})=0$$

which is the equation of a line.

Similarly, we can obtain the equation of line for any signs of $(x+y-5)$ and $(3x-2y+7)$ Thus, point P must move on a line.

Question 21:

Find equation of the line which is equidistant from parallel lines $9x+6y-7=0$ and $3x+2y+6=0$

Answer 21:

The equations of the given lines are

$9x+6y-7=0\ldots$

$3x+2y+6=0$

Let $\mathrm{P}(h,k)$ be the arbitrary point that is equidistant from lines (1) and (2). The perpendicular distance of $\mathrm{P}(h,k)$ from line (1) is given by

$$d_1=\frac{|9h+6k-7|}{(9)^2+(6)^2}=\frac{|9h+6k-7|}{\sqrt{117}}=\frac{|9h+6k-7|}{3\sqrt{13}}$$

The perpendicular distance of $\mathrm{P}(h,k)$ from line (2) is given by

$$d_2=\frac{|3h+2k+6|}{\sqrt{(3)^2+(2)^2}}=\frac{|3h+2k+6|}{\sqrt{13}}$$

Since $\mathrm{P}(h,k)$ is equidistant from lines (1) and (2), $d_1=d_2$

$$\therefore \frac{|9h+6k-7|}{3\sqrt{13}}=\frac{|3h+2k+6|}{\sqrt{13}}$$

$$\Rightarrow |9h+6k-7|=3|3h+2k+6|$$

$$\Rightarrow |9h+6k-7|=\pm 3(3h+2k+6)$$

$\Rightarrow 9h+6k-7=3(3h+2k+6)$ or $9h+6k-7=-3(3h+2k+6)$

The case $9h+6k-7=3(3h+2k+6)$ is not possible as $9h+6k-7=3(3h+2k+6)\Rightarrow -7=18$

(which is absurd)

$\therefore 9h+6k-7=-3(3h+2k+6)$

$9h+6k-7=-9h-6k-18$

$\Rightarrow 18h+12k+11=0$

Thus, the required equation of the line is $18x+12y+11=0$.

Question 22:

A ray of light passing through the point $(1,2)$ reflects on the x-axis at point A and the reflected ray passes through the point $(5,3)$. Find the coordinates of A.

Answer 22:

Let the coordinates of point A be $(a,0)$.

Draw a line (AL) perpendicular to the x-axis.

We know that angle of incidence is equal to angle of reflection. Hence, let

$\angle BAL=\angle CAL=\Phi$

Let $\angle CAX=\theta$

$\therefore \angle OAB=180^{\circ}-(\theta+2\Phi)=180^{\circ}-\left[\theta+2\left(90^{\circ}-\theta\right)\right]$

$=180^{\circ}-\theta-180^{\circ}+2\theta=\theta$

$\therefore \angle BAX=180^{\circ}-\theta$

Now, slope of line $\mathrm{AC}=\dfrac{3-0}{5-a}$

$\Rightarrow \tan\theta=\dfrac{3}{5-a}$

Slope of line $\mathrm{AB}=\dfrac{2-0}{1-a}$

$\Rightarrow \tan\left(180^{\circ}-\theta\right)=\dfrac{2}{1-a}$

$\Rightarrow -\tan\theta=\dfrac{2}{1-a}$

$\Rightarrow \tan\theta = \dfrac{2}{a-1}$

From equations (1) and (2), we obtain

$\dfrac{3}{5-a} = \dfrac{2}{a-1}$

$\Rightarrow 3a-3 = 10-2a$

$\Rightarrow a = \dfrac{13}{5}$

Thus, the coordinates of point A are $\left(\dfrac{13}{5}, 0\right)$.

Question 23:

Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2}, 0\right)$ and $\left(-\sqrt{a^2-b^2}, 0\right)$ to the line $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ is b^2.

Answer 23:

The equation of the given line is

$\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$

Or, $bx\cos\theta + ay\sin\theta - ab = 0$

Length of the perpendicular from point $\left(\sqrt{a^2-b^2}, 0\right)$ to line (1) is

$$p_1 = \frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right) + a\sin\theta(0) - ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{\left|b\cos\theta\sqrt{a^2-b^2} - ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Length of the perpendicular from point $\left(-\sqrt{a^2-b^2}, 0\right)$ to line (2) is

$$p_2 = \frac{\left|b\cos\theta\left(-\sqrt{a^2-b^2}\right) + a\sin\theta(0) - ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{\left|b\cos\theta\sqrt{a^2-b^2} + ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Multiply the equations p_1 and p_2

$$p_1 p_2 = \frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|\left|\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right)^2}$$

$$= \frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}-ab\right)\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}\right)^2-(ab)^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|b^2\cos^2\theta\left(a^2-b^2\right)-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|a^2b^2\cos^2\theta-b^4\cos^2\theta-a^2b^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\sin^2\theta-a^2\cos^2\theta\right|}{b^2\cos^2\theta+a^2\sin^2\theta} \quad \left[\sin^2\theta+\cos^2\theta=1\right]$$

$$= \frac{b^2\left|-\left(b^2\cos^2\theta+a^2\sin^2\theta\right)\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= b^2$$

Hence, proved.

Question 24:

A person standing at the junction (crossing) of two straight paths represented by the equations $2x-3y+4=0$ and $3x+4y-5=0$ wants to reach the path whose equation is $6x-7y+8=0$ in the least time. Find equation of the path that he should follow.

Answer 24:

The equations of the given lines are

$$2x-3y+4=0$$

$3x+4y-5=0$

$6x-7y+8=0$

The person is standing at the junction of the paths represented by lines (1) and (2).

On solving equations (1) and (2), we obtain $x=-\frac{1}{17}$ and $y=\frac{22}{17}$.

Thus, the person is standing at point $\left(-\frac{1}{17},\frac{22}{17}\right)$

The person can reach path (3) in the least time if he walks along the perpendicular line to (3) from point $\left(-\frac{1}{17},\frac{22}{17}\right)$

Slope of the line $(3)=\frac{6}{7}$

Slope of the line perpendicular to line $(3)=-\frac{1}{\left(\frac{6}{7}\right)}=-\frac{7}{6}$

The equation of the line passing through $\left(-\frac{1}{17},\frac{22}{17}\right)$ and having a slope of $-\frac{7}{6}$ is given by

$\left(y-\frac{22}{17}\right)=-\frac{7}{6}\left(x+\frac{1}{17}\right)$

$6(17y-22)=-7(17x+1)$

$102y-132=-119x-7$

$119x+102y=125$

Hence, the path that the person should follow is $119x+102y=125$.