Chapter 11: Conic Sections

Examples

Example 1: Find an equation of the circle with centre at $(0,0)$ and radius r.

Solution: Given that, centre is at $(0,0)$ and radius is r.

So, $h=k=0$.

The general form of the equation of a circle is,

$(x-h)^2+(y-k)^2=r^2$

Substitute the values,

$(x-0)^2+(y-0)^2=r^2$

Therefore, the equation of the circle is $x^2+y^2=r^2$.

Example 2: Find the equation of the circle with centre $(-3,2)$ and radius 4.

Solution: Given that, centre is at $(-3,2)$ and radius is 4.

So, $h=-3, k=2$ and $r=4$.

The general form of the equation of a circle is,

$(x-h)^2+(y-k)^2=r^2$

Substitute the values,

$(x--3)^2+(y-2)^2=4^2$

That is, $(x+3)^2+(y-2)^2=16$

Therefore, the equation of the required circle is

$(x+3)^2+(y-2)^2=16$

Example 3: Find the centre and the radius of the circle $x^2+y^2+8x+10y-8=0$.

Solution: Given that, the equation is $x^2+y^2+8x+10y-8=0$.

It can be written as,

$\left(x^2+8x\right)+\left(y^2+10y\right)=8$

Now, completing the squares within the parenthesis, we get

$\left(x^2+8x+16\right)+\left(y^2+10y+25\right)=8+16+25$

That is, $(x+4)^2+(y+5)^2=49$

Rewrite the equation.

$\{x-(-4)\}^2+\{y-(-5)\}^2=7^2$

Therefore, the given circle has centre at $(-4,-5)$ and radius 7.

Example 4: Find the equation of the circle which passes through the points $(2,-2)$, and $(3,4)$ and whose centre lies on the line $x+y=2$.

Solution: The general form of the equation of a circle is,

$(x-h)^2+(y-k)^2=r^2$.

Here, the circle passes through $(2,-2)$ and $(3,4)$.

Now, $(2-h)^2+(-2-k)^2=r^2 \to (1)$

and $(3-h)^2+(4-k)^2=r^2 \to (2)$

Also here, the centre lies on the line $x+y=2$, we have

$h+k=2$

When solving the equations $(1),(2)$ and (3), we get

$h=0.7, k=1.3$ and $r^2=12.58$

Therefore, the equation of the required circle is $(x-0.7)^2+(y-1.3)^2=12.58$.

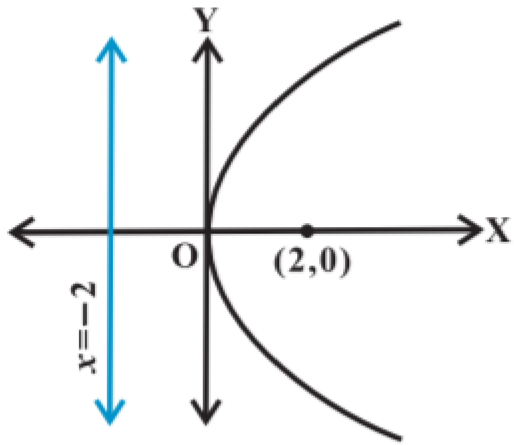
Example 5: Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2=8x$.

Solution: Here, the equation involves y^2, that means the axis of symmetry is along the x-axis.

The coefficient of x is positive.

Thus, the parabola opens to the right.

When comparing the given equation with $y^2=4ax$, we get $a=2$.



Then, the focus of the parabola is $(2,0)$ and the equation of the directrix of the parabola is $x=-2$.

Length of the latus rectum is,

$4a=4\times 2=8$.

Example 6: Find the equation of the parabola with focus $(2,0)$ and directrix $x=-2$.

Solution: Here, the focus is $(2,0)$ and it lies on the x-axis.

So, the x-axis itself is the axis of the parabola.

Thus, the equation of the parabola will be of the form either $y^2=4ax$ or $y^2=-4ax$.

Given that, the directrix is $x=-2$ and the focus is $(2,0)$.

Then, the parabola is of the form $y^2=4ax$ with $a=2$.

Now, the required equation is,

$y^2=4(2)x$

Multiply the terms,

$y^2=8x$

Therefore, the equation of the parabola with focus $(2,0)$ and directrix $x=-2$ is $y^2=8x$.

Example 7: Find the equation of the parabola with vertex at $(0,0)$ and focus at $(0,2)$.

Solution: Here, the vertex is at $(0,0)$ and the focus is at $(0,2)$ which lies on y-axis.

So, the y-axis is the axis of the parabola.

Thus, equation of the parabola is of the form $x^2=4ay$.

Now, $x^2 = 4(2)y$

Multiply the terms,

$x^2 = 8y$

Therefore, the equation of the parabola with vertex at $(0,0)$ and focus at $(0,2)$ is $x^2 = 8y$.

Example 8: Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $(2,-3)$.

Solution: Here, the parabola is symmetric about y-axis and has its vertex at the origin.

So, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$.

However, the parabola passes through $(2,-3)$ which lies in the fourth quadrant.

So, the parabola must open downwards.

Then, the equation is of the form $x^2 = -4ay$.

Given that the parabola passes through $(2,-3)$, we have

$$2^2 = -4a(-3) \Rightarrow a = \frac{1}{3}$$

Thus, the equation of the parabola is

$$x^2 = -4\left(\frac{1}{3}\right)y$$

Multiply both sides by 3,

$3x^2 = -4y$

Therefore, the equation of the parabola which is symmetric about the y-axis, and passes through the point $(2,-3)$ is $3x^2 = -4y$.

Example 9: Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Solution: Given that, the equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

This can be written as,

$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$$

It can be observed that the denominator of $\frac{x^2}{5^2}$ is greater than the denominator of $\frac{y^2}{3^2}$.

Thus, the major axis is along the x-axis and the minor axis is along the y-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a=5$ and $b=3$.

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$$c=\sqrt{25-9}$$

Subtract the numbers,

$$c=\sqrt{16}=4$$

Therefore, The coordinates of the foci are $(4,0)$ and $(-4,0)$.

The coordinates of the vertices are $(5,0)$ and $(-5,0)$

Length of major axis $=2a=10$

Length of minor axis $=2b=6$

Eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$$e=\frac{4}{5}$$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2\times 9}{5}=\frac{18}{5}.$$

Question 10. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis and the eccentricity of the ellipse $9x^2+4y^2=36$.

Solution: Given that, the equation is $9x^2 + 4y^2 = 36$.

It can be written as

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Or, $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \rightarrow (1)$

It can be observed that the denominator of $\frac{y^2}{3^2}$ is greater than the denominator of $\frac{x^2}{2^2}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing equation (1) with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get $b = 2$ and $a = 3$.

Now, $c = \sqrt{a^2 - b^2}$

Substitute the values,

$$c = \sqrt{9 - 4}$$

Subtract the numbers,

$$c = \sqrt{5}$$

Therefore, The coordinates of the foci are $\left(0, \sqrt{5}\right)$ and $\left(0, -\sqrt{5}\right)$

The coordinates of the vertices are $(0, 3)$ and $(0, -3)$

Length of major axis $= 2a = 6$

Length of minor axis $= 2b = 4$

Eccentricity is,

$$e = \frac{c}{a}$$

Substitute the values,

$$e = \frac{\sqrt{5}}{3}$$

Example 11: Find the equation of the ellipse whose vertices are $(\pm 13, 0)$ and foci are $(\pm 5, 0)$.

Solution: Given that the coordinates of vertices are $(\pm 13, 0)$ and the coordinates of foci are $(\pm 5, 0)$.

Here, the vertices are on the x-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $a=13$ and $c=5$.

It is known that $a^2=b^2+c^2$

That is, $13^2=b^2+5^2$

Determine the squares,

$169=b^2+25$

Subtract 25 from both sides,

$b^2=169-25$

Then, $b=\sqrt{144}=12$

Thus, the equation of the ellipse is $\frac{x^2}{13^2}+\frac{y^2}{12^2}=1$ or $\frac{x^2}{169}+\frac{y^2}{144}=1$.

Example 12: Find the equation of the ellipse, whose length of the major axis is 20 and foci are $(0,\pm5)$.

Solution: Given that, Length of major axis $=20$ and foci $=(0,\pm5)$.

Here, the foci are on the y-axis.

So, the major axis is along the y-axis.

Therefore, the equation of the ellipse will be of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where a is the semi-major axis.

From the given data, $2a=20 \Rightarrow a=10$ and $c=5$.

It is known that $a^2=b^2+c^2$

Then, $10^2=b^2+5^2$

Determine the squares,

$100=b^2+25$

Subtract 25 from both sides,

$b^2=100-25$

$b=\sqrt{75}$

Therefore, the equation of the ellipse is $\frac{x^2}{\left(\sqrt{75}\right)^2}+\frac{y^2}{10^2}=1$ or $\frac{x^2}{75}+\frac{y^2}{100}=1$.

Example 13: Find the equation of the ellipse, with major axis along the x-axis and passing through the points $(4,3)$ and $(-1,4)$.

Solution: Here, the major axis is on the x-axis.

So, the equation of the ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to (1)$

Where, a is the semi-major axis.

Given that, the ellipse passes through points $(4,3)$ and $(-1,4)$.

Thus, $\frac{16}{a^2}+\frac{9}{b^2}=1 \to (2)$

$$\frac{1}{a^2}+\frac{16}{b^2}=1 \to (3)$$

Then, $\frac{1}{a^2}=1-\frac{16}{b^2}=\frac{b^2-16}{b^2}$

Substitute this in (2),

$$16\left(\frac{b^2-16}{b^2}\right)+\frac{9}{b^2}=1$$

Open brackets,

$$\frac{16b^2-256}{b^2}+\frac{9}{b^2}=1$$

Add the terms,

$$\frac{16b^2-256+9}{b^2}=1$$

Add the numbers,

$$\frac{16b^2-247}{b^2}=1$$

$$16b^2-247=b^2$$

$$15b^2=247$$

$b^2 = \frac{247}{15}$

Substitute the value of b^2 in (3),

$$\frac{1}{a^2} + \frac{16}{\frac{247}{15}} = 1$$

That is, $\frac{1}{a^2} + \frac{16\times15}{247} = 1$

$$\frac{1}{a^2} = 1 - \frac{16\times15}{247}$$

Multiply the numbers,

$$\frac{1}{a^2} = 1 - \frac{240}{247}$$

Take LCM,

$$\frac{1}{a^2} = \frac{7}{247}$$

Then, $a^2 = \frac{247}{7}$

Substitute the values of a^2 and b^2 in (1) to get the equation of ellipse.

Therefore, the equation of the ellipse is $\frac{x^2}{\left(\frac{247}{7}\right)} + \frac{y^2}{\frac{247}{15}} = 1$ or $7x^2 + 15y^2 = 247$.

Example 14: Find the coordinates of the foci and the vertices, the eccentricity, the length of the latus rectum of the hyperbolas:

(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$,

(ii) $y^2 - 16x^2 = 16$

Solution:

(i) Given that, the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

When comparing the equation $\frac{x^2}{9}-\frac{y^2}{16}=1$ with the standard equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a=3$ and $b=4$

Now, $c=\sqrt{a^2+b^2}$

Substitute thew values,

$$c=\sqrt{9+16}$$

Add the numbers and simplify,

$$c=5$$

Thus, the coordinates of the foci are $(\pm 5,0)$ and that of vertices are $(\pm 3,0)$.

Now, The eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$$e=\frac{5}{3}$$

The length latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2\times 16}{3}=\frac{32}{3}$$

(ii) Given that, the equation is $y^2-16x^2=16$.

Divide the equation by 16 on both sides,

$$\frac{y^2}{16}-\frac{x^2}{1}=1$$

When comparing the equation with the standard equation of hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we get

$a=4$ and $b=1$

Now, $c=\sqrt{a^2+b^2}$

Substitute the values,

$$c=\sqrt{16+1}$$

Add the numbers,

$c=\sqrt{17}$

Thus, the coordinates of the foci are $(0,\pm\sqrt{17})$ and that of the vertices are $(0,\pm 4)$.

The eccentricity is,

$e=\frac{c}{a}$

Substitute the values,

$e=\frac{\sqrt{17}}{4}$

The length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$\frac{2\times 1}{2}=\frac{1}{2}$.

Question 15. Find the equation of the hyperbola with foci $(0,\pm 3)$ and vertices $\left(0,\pm\frac{\sqrt{11}}{2}\right)$

Solution: Given that, Vertices $\left(0,\pm\frac{\sqrt{11}}{2}\right)$ and foci $(0,\pm 3)$.

Here, the vertices are on the y-axis.

Then, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Given the vertices are $\left(0,\pm\frac{\sqrt{11}}{2}\right)$

That is, $a=\frac{\sqrt{11}}{2}$

Since the foci are $(0,\pm 3)$

$c=3$

It is known that $a^2+b^2=c^2$

Then, $\left(\frac{\sqrt{11}}{2}\right)^2+b^2=3^2$

$b^2 = 9 - \frac{11}{4} = \frac{25}{4}$

Therefore, the equation of the hyperbola is $\frac{y^2}{\left(\frac{11}{4}\right)} - \frac{x^2}{\left(\frac{25}{4}\right)} = 1$ or $100y^2 - 44x^2 = 275.$

Question 16. Find the equation of the hyperbola where foci are $(0, \pm 12)$ and the length of latus rectum is 36.

Solution: Given that, Foci are $(0, \pm 12)$.

Then, $c = 4$ and it is also given that the latus rectum is of length 36 .

Here, the foci are on the y-axis.

Then, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Length of the latus rectum $= \frac{2b^2}{a} = 36$ or $b^2 = 18a$

It is known that, $c^2 = a^2 + b^2$

Substitute the values,

$144 = a^2 + 18a$

Subtract 144 from both sides,

$a^2 + 18a - 144 = 0 \Rightarrow a = -24, 6.$

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$ or $3y^2 - x^2 = 108$.

Example 17: The focus of a parabolic mirror as shown in Fig 11.33 is at a distance of 5cm from its vertex. If the mirror is 45cm deep, find the distance AB (Fig 11.33).

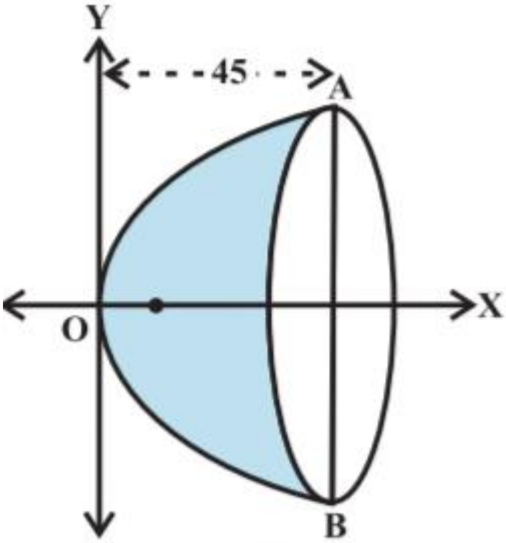


Fig 11.33

Solution: Given that, the distance from the focus to the vertex is 5cm.

Then, $a = 5$.

If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is,

$y^2 = 4(5)x$

Multiply the numbers,

$y^2 = 20x$

It is given that, $x = 45$

Substitute the value of x,

$y^2 = 900$

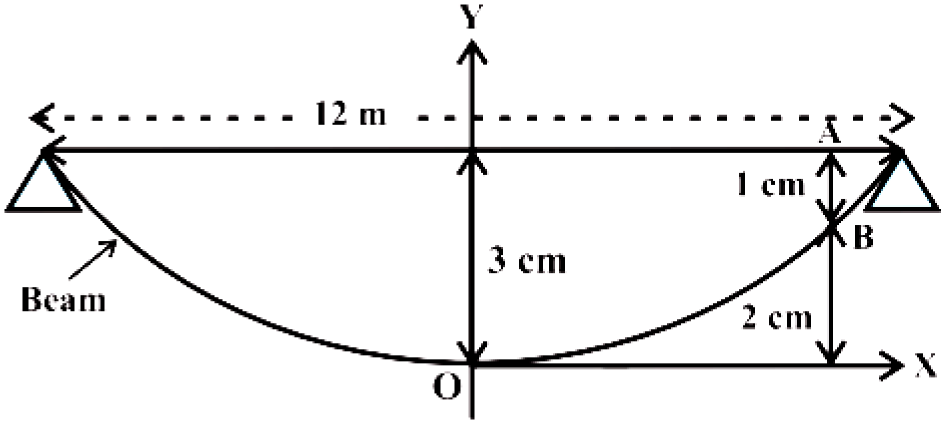
$y = \pm 30$

Therefore, $\text{AB} = 2y = 2 \times 30 = 60\text{cm}$.

Example 18: A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1cm ?

Solution: Let the vertex be at the lowest point and the axis vertical.

Let the coordinate axis be chosen as shown in Figure,



Here, the equation of the parabola is form $x^2 = 4ay$.

And it passes through $\left(6, \frac{3}{100}\right)$.

Then we have $(6)^2 = 4a\left(\frac{3}{100}\right)$

That is, $a = \frac{36\times 100}{12} = 300\,\text{m}$

Let AB be the deflection of the beam which is $\frac{1}{100}\,\text{m}$.

The coordinates of B are $\left(x, \frac{2}{100}\right)$.

Thus, $x^2 = 4\times 300\times\frac{2}{100} = 24$.

Then, $x = \sqrt{24}$

Simplify the radicand,

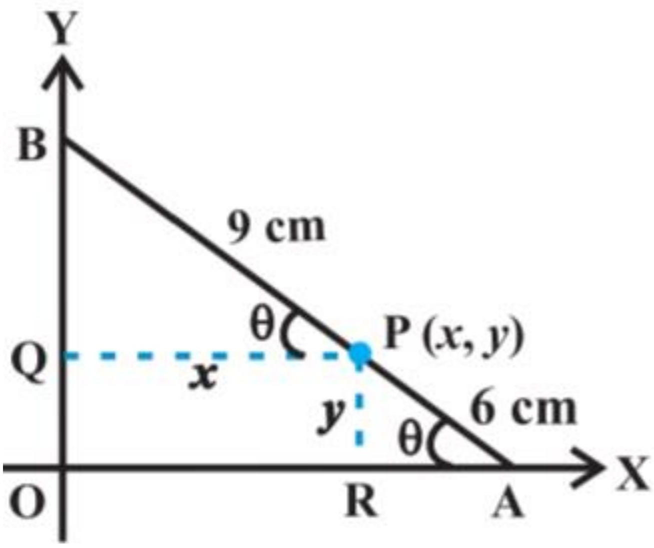
$x = 2\sqrt{6}$ metres.

Example 19: A rod AB of length 15cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point $\text{P}(x, y)$ is taken on the rod in such a way that $\text{AP} = 6\text{cm}$. Show that the locus of P is an ellipse.

Solution: Let AB be the rod making an angle θ with OX as shown in figure and $\text{P}(x, y)$ the point on it such that $\text{AP} = 6\,\text{cm}$.

Here, $\text{AB} = 15\,\text{cm}$, then $\text{PB} = 9\,\text{cm}$.

From P draw PQ and PR perpendiculars on y-axis and x-axis, respectively.



From $\Delta PBQ, \cos\theta = \frac{x}{9}$

From Δ PRA, $\sin\theta = \frac{y}{6}$

It is known that $\cos^2\theta + \sin^2\theta = 1$.

That is, $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

or $\frac{x^2}{81} + \frac{y^2}{36} = 1$

Therefore, the locus of P is an ellipse.

Exercise 11.1

Question 1. Find the equation of the circle with centre $(0,2)$ and radius 2.

Solution: It is known that, the equation of a circle with centre (h,k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$

Here, centre $(h,k) = (0,2)$ and radius $(r) = 2$.

Now, the equation of the circle is,

$(x-0)^2 + (y-2)^2 = 2^2$

Apply the perfect square formula: $(a-b)^2 = a^2 - 2ab + b^2$

$x^2+y^2+4-4y-4=0$

Subtract the numbers,

$x^2+y^2-4y=0$

Therefore, the equation of the circle is $x^2+y^2-4y=0$.

Question 2. Find the equation of the circle with centre $(-2,3)$ and radius 4.

Solution: It is known that, the equation of a circle with centre (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$

Here, centre $(h,k)=(-2,3)$ and radius $(r)=4$.

Now, the equation of the circle is,

$(x+2)^2+(y-3)^2=(4)^2$

Apply the perfect square formula: $(a-b)^2=a^2-2ab+b^2$

$x^2+4x+4+y^2-6y+9=16$

Subtract 16 from both sides,

$x^2+y^2+4x-6y-3=0$

Therefore, the equation of the circle is $x^2+y^2+4x-6y-3=0$.

Question 3. Find the equation of the circle with centre $\left(\frac{1}{2},\frac{1}{4}\right)$ and radius $\frac{1}{12}$ Answer 3:

Solution: It is known that, the equation of a circle with centre (h,k) and radius r is

$(x-h)^2+(y-k)^2=r^2$

Here, centre $(h,k)=\left(\frac{1}{2},\frac{1}{4}\right)$ and radius $(r)=\frac{1}{12}$

Now, the equation of the circle is

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{4}\right)^2=\left(\frac{1}{12}\right)^2$$

Apply the perfect square formula: $(a-b)^2=a^2-2ab+b^2$

$$x^2-x+\frac{1}{4}+y^2-\frac{y}{2}+\frac{1}{16}=\frac{1}{144}$$

Subtract $\frac{1}{144}$ from both sides,

$$x^2-x+\frac{1}{4}+y^2-\frac{y}{2}+\frac{1}{16}-\frac{1}{144}=0$$

$$144x^2-144x+36+144y^2-72y+9-1=0$$

Add and subtract the numbers,

$$144x^2-144x+144y^2-72y+44=0$$

Divide both sides by 4,

$$36x^2-36x+36y^2-18y+11=0$$

Rewrite the equation,

$$36x^2+36y^2-36x-18y+11=0$$

Therefore, the equation of the circle is $36x^2+36y^2-36x-18y+11=0$.

Question 4. Find the equation of the circle with centre \$(1,1)\$ and radius \$\sqrt{2}\$.

Solution: It is known that, the equation of a circle with centre (h,k) and radius r is given by, $(x-h)^2+(y-k)^2=r^2$

Here, centre $(h,k)=(1,1)$ and radius $(r)=\sqrt{2}$.

Now, the equation of the circle is,

$$(x-1)^2+(y-1)^2=(\sqrt{2})^2$$

Apply the perfect square formula: $(a-b)^2=a^2-2ab+b^2$

$$x^2-2x+1+y^2-2y+1=2$$

Add the numbers,

$$x^2-2x+y^2-2y+2=2$$

Subtract 2 from both sides and rewrite,

$$x^2+y^2-2x-2y=0$$

Therefore, the equation of the circle is $x^2+y^2-2x-2y=0$.

Question 5. Find the equation of the circle with centre $(-a,-b)$ and radius $\sqrt{a^2-b^2}$.

Solution: It is known that, the equation of a circle with centre (h,k) and radius r is given by, $(x-h)^2+(y-k)^2=r^2$

Here, centre $(h,k)=(-a,-b)$ and radius $(r)=\sqrt{a^2-b^2}$.

Now, the equation of the circle is,

$$(x+a)^2+(y+b)^2=\left(\sqrt{a^2-b^2}\right)^2$$

Apply the perfect square formula: $(a+b)^2=a^2+2ab+b^2$

$$x^2+2ax+a^2+y^2+2by+b^2=a^2-b^2$$

Add $-\left(a^2+b^2\right)$ to both sides and rewrite the equation,

$$x^2+y^2+2ax+2by+2b^2=0$$

Therefore, the equation of the circle is $x^2+y^2+2ax+2by+2b^2=0$.

Question 6. Find the centre and radius of the circle $(x+5)^2+(y-3)^2=36$.

Solution: Here, the equation of the given circle is $(x+5)^2+(y-3)^2=36$.

Rewrite the equation as,

$$(x+5)^2+(y-3)^2=36 \Rightarrow \{x-(-5)\}^2+(y-3)^2=6^2$$

It is of the form $(x-h)^2+(y-k)^2=r^2$, where $h=-5, k=3$, and $r=6$,

Therefore, the centre of the given circle is $(-5,3)$ and its radius is 6.

Question 7. Find the centre and radius of the circle $x^2+y^2-4x-8y-45=0$

Solution: Here, the equation of the given circle is $x^2+y^2-4x-8y-45=0$.

Rewrite the equation,

$$x^2+y^2-4x-8y-45=0 \Rightarrow \left(x^2-4x\right)+\left(y^2-8y\right)=45$$

Add $\left(2^2+4^2\right)$ to the left side and subtract $\left(2^2+4^2\right)$ from left side,

$$\left\{x^2-2(x)(2)+2^2\right\}+\left\{y^2-2(y)(4)+4^2\right\}-4-16=45$$

We have, $(a-b)^2=a^2-2ab+b^2$,

$(x-2)^2+(y-4)^2=65$

Rewrite the equation as,

$(x-2)^2+(y-4)^2=\left(\sqrt{65}\right)^2$

It is of the form $(x-h)^2+(y-k)^2=r^2$, where $h=2, k=4$ and $r=\sqrt{65}$.

Therefore, the centre of the given circle is $(2,4)$ and its radius is $\sqrt{65}$.

Question 8. Find the centre and radius of the circle $x^2+y^2-8x+10y-12=0$

Solution: Here, the equation of the given circle is $x^2+y^2-8x+10y-12=0$.

Rewrite the equation,

$\left(x^2-8x\right)+\left(y^2+10y\right)=12$

Add $\left(4^2+5^2\right)$ to the left side and subtract $\left(4^2+5^2\right)$ from left side,

$\left\{x^2-2(x)(4)+4^2\right\}+\left\{y^2+2(y)(5)+5^2\right\}-16-25=12$

We have, $(a-b)^2=a^2-2ab+b^2$,

$(x-4)^2+(y+5)^2=53$

Rewrite the equation as,

$(x-4)^2+\{y-(-5)\}^2=\left(\sqrt{53}\right)^2$

It is of the form $(x-h)^2+(y-k)^2=r^2$, where $h=4, k=-5$ and $r=\sqrt{53}$.

Therefore, the centre of the given circle is $(4,-5)$ and its radius is $\sqrt{53}$.

Question 9. Find the centre and radius of the circle \$2 x^{2}+2 y^{2}-x=0\$

Solution: Here, the equation of the given circle is $2x^2+2y^2-x=0$.

Rewrite the equation,

$\left(2x^2-x\right)+2y^2=0$

Take 2 out side,

$2\left[\left(x^2-\frac{x}{2}\right)+y^2\right]=0$

Rewrite $\frac{x}{2}$ then add $\left(\frac{1}{4}\right)^2$ to the left side and subtract $\left(\frac{1}{4}\right)^2$ from left side,

$$\left\{x^2-2\cdot x\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2\right\}+y^2-\left(\frac{1}{4}\right)^2=0$$

We have, $(a-b)^2=a^2-2ab+b^2$,

$$\left(x-\frac{1}{4}\right)^2+(y-0)^2=\left(\frac{1}{4}\right)^2$$

It is of the form $(x-h)^2+(y-k)^2=r^2$, where $h=\frac{1}{4},k=0$ and $r=\frac{1}{4}$.

Therefore, the centre of the given circle is $\left(\frac{1}{4},0\right)$ and its radius is $\frac{1}{4}$.

Question 10. Find the equation of the circle passing through the points $(4,1)$ and $(6,5)$ and whose centre is on the line $4x+y=16$

Solution: Let the equation of the required circle be $(x-h)^2+(y-k)^2=r^2$.

Here, the circle passes through points $(4,1)$ and $(6,5)$,

So, $(4-h)^2+(1-k)^2=r^2\rightarrow(1)$

$(6-h)^2+(5-k)^2=r^2\rightarrow(2)$

Since the centre (h,k) of the circle lies on line $4x+y=16$,

$4h+k=16\rightarrow(3)$

From equations (1) and (2),

$(4-h)^2+(1-k)^2=(6-h)^2+(5-k)^2$

Open the brackets,

$16-8h+h^2+1-2k+k^2=36-12h+h^2+25-10k+k^2$

Cancel the common terms,

$16-8h+1-2k=36-12h+25-10k$

Combine like terms,

$4h+8k=44$

Divide both sides by 2,

$h+2k=11 \to (4)$

On solving equations (3) and (4), we get $h=3$ and $k=4$.

Substitute the values of h and k in equation (1),

$(4-3)^2+(1-4)^2=r^2$

Subtract the numbers,

$(1)^2+(-3)^2=r^2$

Determine the squares,

$1+9=r^2$

Add the numbers,

$r^2=10$

Square root both sides,

$r=\sqrt{10}$

Thus, the equation of the required circle is,

$(x-3)^2+(y-4)^2=\left(\sqrt{10}\right)^2$

Open brackets,

$x^2-6x+9+y^2-8y+16=10$

Add the numbers,

$x^2+y^2-6x-8y+15=0$

Therefore, the equation of the circle passing through the points $(4,1)$ and $(6,5)$, whose centre is on the line $4x+y=16$ is $x^2+y^2-6x-8y+15=0$.

Question 11: Find the equation of the circle passing through the points $(2,3)$ and $(-1,1)$, and whose centre is on the line $x-3y-11=0$.

Solution: Let the equation of the required circle be $(x-h)^2+(y-k)^2=r^2$.

Here, the circle passes through points $(2,3)$ and $(-1,1)$.

So, $(2-h)^2+(3-k)^2=r^2 \to (1)$

$(-1-h)^2+(1-k)^2=r^2 \to (2)$

Since the centre (h,k) of the circle lies on line $x-3y-11=0$,

$h-3k=11 \to (3)$

From equations (1) and (2), we get

$(2-h)^2+(3-k)^2=(-1-h)^2+(1-k)^2$

Open the brackets,

$4-4h+h^2+9-6k+k^2=1+2h+h^2+1-2k+k^2$

Cancel the common terms,

$4-4h+9-6k=1+2h+1-2k$

Add the numbers,

$6h+4k=11 \to (4)$

On solving equations (3) and (4), we get $h=\frac{7}{2}$ and $k=\frac{-5}{2}$.

Substitute the values of h and k in equation (1),

$$\left(2-\frac{7}{2}\right)^2+\left(3+\frac{5}{2}\right)^2=r^2 \Rightarrow \left(\frac{4-7}{2}\right)^2+\left(\frac{6+5}{2}\right)^2=r^2$$

Subtract the numbers,

$$\left(\frac{-3}{2}\right)^2+\left(\frac{11}{2}\right)^2=r^2$$

Determine the squares,

$$\frac{9}{4}+\frac{121}{4}=r^2$$

Add the numbers,

$$\frac{130}{4}=r^2$$

Now, the equation of the required circle is,

$$\left(x-\frac{7}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=\frac{130}{4} \Rightarrow \left(\frac{2x-7}{2}\right)^2+\left(\frac{2y+5}{2}\right)^2=\frac{130}{4}$$

Subtract the terms,

$4x^2-28x+49+4y^2+20y+25=130$

Add the numbers,

$4x^2+4y^2-28x+20y-56=0$

Take 4 out side,

$$4\left(x^2+y^2-7x+5y-14\right)=0$$

Divide both sides by 4,

$$x^2+y^2-7x+5y-14=0$$

The equation of the circle passing through the points $(2,3)$ and $(-1,1)$,whose centre is on the line $x-3y-11=0$ is $x^2+y^2-7x+5y-14=0$.

Question 12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $(2,3)$.

Solution: Let the equation of the required circle be $(x-h)^2+(y-k)^2=r^2$.

Here, the radius of the circle is 5 and its centre lies on the x-axis,

Then, $k=0$ and $r=5$.

Now, the equation of the circle becomes $(x-h)^2+y^2=25$.

Given that the circle passes through point $(2,3)$.

Thus, $(2-h)^2+3^2=25$

Subtract 3^2 from both sides,

$$(2-h)^2=25-9$$

Subtract the numbers,

$$(2-h)^2=16$$

Then, $2-h=\pm\sqrt{16}=\pm4$

If $2-h=4$, then $h=-2$

If $2-h=-4$, then $h=6$

When $h=-2$, the equation of the circle becomes

$$(x+2)^2+y^2=25$$

We have, $(a+b)^2=a^2+2ab+b^2$,

$$x^2+4x+4+y^2=25$$

Subtract 25 from both sides,

$$x^2+y^2+4x-21=0$$

When $h=6$, the equation of the circle is,

$$(x-6)^2+y^2=25$$

We have, $(a+b)^2=a^2+2ab+b^2$,

$$x^2-12x+36+y^2=25$$

Subtract 25 from both sides,

$$x^2+y^2-12x+11=0$$

Question 13. Find the equation of the circle passing through $(0,0)$ and making intercepts a and b on the coordinate axes.

Solution: Let the equation of the required circle be $(x-h)^2+(y-k)^2=r^2$.

Here, the centre of the circle passes through $(0,0)$.

Then, $(0-h)^2+(0-k)^2=r^2$

Subtract the terms,

$$h^2+k^2=r^2$$

The equation of the circle now becomes $(x-h)^2+(y-k)^2=h^2+k^2$.

Given that, the circle makes intercepts a and b on the coordinate axes.

This means that the circle passes through points $(a,0)$ and $(0,b)$.

Thus,

$$(a-h)^2+(0-k)^2=h^2+k^2 \rightarrow (1)$$

$$(0-h)^2+(b-k)^2=h^2+k^2 \rightarrow (2)$$

From equation (1), we get $a^2-2ah+h^2+k^2=h^2+k^2$.

Cancel the common terms,

$$a^2-2ah=0$$

Take a out side,

$$a(a-2h)=0$$

By zero product property,

$a=0$ or $(a-2h)=0$

But, $a \neq 0$; hence, $(a-2h)=0 \Rightarrow h=\frac{a}{2}$

From equation (2), we get $h^2+b^2-2bk+k^2=h^2+k^2$

Cancel the common terms,

$b^2-2bk=0$

Take b out side,

$b(b-2k)=0$

By zero product property,

$b=0$ or $(b-2k)=0$

But, $b \neq 0$; hence, $(b-2k)=0 \Rightarrow k=\frac{b}{2}$

Thus, the equation of the required circle is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2$$

Take LCM,

$$\left(\frac{2x-a}{2}\right)^2+\left(\frac{2y-b}{2}\right)^2=\frac{a^2+b^2}{4}$$

We have, $(a-b)^2=a^2-2ab+b^2$,

$4x^2-4ax+a^2+4y^2-4by+b^2=a^2+b^2$

Cancel the common terms,

$4x^2+4y^2-4ax-4by=0$

Divide both sides by 4,

$x^2+y^2-ax-by=0$

Therefore, the equation of the circle passing through $(0,0)$ and making intercepts a and b on the coordinate axes is $x^2+y^2-ax-by=0$.

Question 14. Find the equation of a circle with centre $(2,2)$ and passes through the point $(4,5)$.

Solution: Given that, the centre of the circle is $(h,k)=(2,2)$.

The circle passes through point $(4,5)$.

So, the radius (r) of the circle is the distance between the points $(2,2)$ and $(4,5)$.

By distance formula,

$$r=\sqrt{(2-4)^2+(2-5)^2}$$

Subtract the numbers,

$$r=\sqrt{(-2)^2+(-3)^2}$$

Determine the squares,

$$r=\sqrt{4+9}$$

Add the numbers,

$$r=\sqrt{13}$$

Now, the equation of the circle is,

$$(x-h)^2+(y-k)^2=r^2$$

Substitute the known values,

$$(x-2)^2+(y-2)^2=(\sqrt{13})^2$$

We have, $(a-b)^2=a^2-2ab+b^2$,

$$x^2-4x+4+y^2-4y+4=13$$

Combine the like terms,

$$x^2+y^2-4x-4y-5=0$$

Therefore, the equation of a circle with centre $(2,2)$ and passes through the point $(4,5)$ is $x^2+y^2-4x-4y-5=0$.

Question 15. Does the point $(-2.5,3.5)$ lie inside, outside or on the circle $x^2+y^2=25$?

Solution: Given that, the equation of the circle is $x^2+y^2=25$.

$$x^2+y^2=25$$

This can be written as,

$$(x-0)^2+(y-0)^2=5^2$$

It is of the form $(x-h)^2+(y-k)^2=r^2$, where $h=0,k=0$, and $r=5$.

Now, Centre $=(0,0)$ and radius $=5$

Distance between point $(-2.5, 3.5)$ and centre $(0, 0)$ is,

$$\sqrt{(-2.5-0)^2+(3.5-0)^2}$$

Subtract the numbers and determine the squares,

$$\sqrt{6.25+12.25}$$

Add the numbers,

$$\sqrt{18.5}$$

Calculate the value,

4.3 (approx.) < 5

Here, the distance between point $(-2.5, 3.5)$ and centre $(0, 0)$ of the circle is less than the radius of the circle.

Therefore, the point $(-2.5, 3.5)$ lies inside the circle.

Exercise 11.2

Question 1. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = 12x$.

Solution: Given that, equation is $y^2 = 12x$.

Here, the coefficient of x is positive.

Thus, the parabola opens towards the right.

When comparing this equation with $y^2 = 4ax$, we get

$$4a = 12 \Rightarrow a = 3$$

Now, Coordinates of the focus $= (a, 0)$

That is $(3, 0)$.

The given equation involves y^2.

Then, the axis of the parabola is the x-axis.

Equation of directrix is,

$$x = -a$$

That is, $x = -3$

Add 3 to both sides,

$$x + 3 = 0$$

Now, length of latus rectum $= 4a$

Substitute the values,

$4 \times 3 = 12$

Question 2. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = 6y$.

Solution: Given that, the equation is $x^2 = 6y$.

Here, the coefficient of y is positive.

So, the parabola opens upwards.

When comparing this equation with $x^2 = 4ay$, we get

$$4a = 6 \Rightarrow a = \frac{3}{2}$$

Thus, Coordinates of the focus $= (0, a) = \left(0, \frac{3}{2}\right)$

The given equation involves x^2.

So, the axis of the parabola is the y-axis.

Now, Equation of directrix is $y = -a$

That is, $y = -\frac{3}{2}$

Length of latus rectum $= 4a = 6$

Question 3. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = -8x$

Solution: Given that, the equation is $y^2 = -8x$.

Here, the coefficient of x is negative.

So, the parabola opens towards the left.

When comparing this equation with $y^2 = -4ax$, we get

$$-4a = -8 \Rightarrow a = 2$$

Thus, Coordinates of the focus $= (-a, 0) = (-2, 0)$

Here, the given equation involves y^2.

So, the axis of the parabola is the x-axis.

Now, Equation of directrix is $x = a$

That is, $x = 2$

Length of latus rectum $= 4a = 8$

Question 4. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = -16y$

Solution: Given that, the equation is $x^2 = -16y$.

Here, the coefficient of y is negative.

So, the parabola opens downwards.

When comparing this equation with $x^2 = -4ay$, we get

$-4a = -16 \Rightarrow a = 4$

Thus, Coordinates of the focus $= (0, -a) = (0, -4)$

Here, the given equation involves x^2.

So, the axis of the parabola is the y-axis.

Now, Equation of directrix is $y = a$.

That is, $y = 4$

Length of latus rectum $= 4a = 16$

Question 5. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = 10x$.

Solution: Given that, the equation is $y^2 = 10x$.

Here, the coefficient of x is positive.

Thus, the parabola opens towards the right.

When comparing this equation with $y^2 = 4ax$, we get

$$4a = 10 \Rightarrow a = \frac{5}{2}$$

Thus, Coordinates of the focus $= (a, 0) = \left(\frac{5}{2}, 0\right)$

Here, the equation involves y^2.

So, the axis of the parabola is the x-axis.

Now, Equation of directrix is $x = -a$.

That is, $x = -\frac{5}{2}$

Length of latus rectum $= 4a = 10$

Question 6. Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = -9y$

Solution: Given that, the equation is $x^2 = -9y$.

Here, the coefficient of y is negative.

So, the parabola opens downwards.

When comparing this equation with $x^2 = -4ay$, we get

$$-4a = -9 \Rightarrow b = \frac{9}{4}$$

Thus, Coordinates of the focus $= (0, -a) = \left(0, -\frac{9}{4}\right)$

Here, the equation involves x^2.

So, the axis of the parabola is the y-axis.

Now, Equation of directrix is $y = a$.

That is $y = \frac{9}{4}$

Length of latus rectum $= 4a = 9$

Question 7. Find the equation of the parabola that satisfies the following conditions: Focus $(6, 0)$; directrix $x = -6$

Solution: Given that, Focus is $(6, 0)$ and directrix $x = -6$.

Here, the focus lies on the x-axis.

So, the x-axis is the axis of the parabola.

Thus, the equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = -4ax$.

It is also observed that the directrix is $x = -6$ which is to the left of the y-axis, while the focus $(6, 0)$ is to the right of the y-axis.

Thus, the parabola is of the form $y^2 = 4ax$.

Here, $a = 6$

Therefore, the equation of the parabola is $y^2 = 24x$.

Question 8. Find the equation of the parabola that satisfies the following conditions: Focus $(0, -3)$; directrix $y = 3$.

Solution: Given that, Focus $= (0, -3)$ and directrix is $y = 3$.

Here, the focus lies on the y-axis.

So, the y-axis is the axis of the parabola.

Thus, the equation of the parabola is either of the form $x^2 = 4$ ay or $x^2 = -4ay$.

It is also observed that the directrix, $y = 3$ is above the x-axis, while the focus $(0, -3)$ is below the x-axis.

So, the parabola is of the form $x^2 = -4ay$.

Here, $a = 3$

Therefore, the equation of the parabola is $x^2 = -12y$.

Question 9. Find the equation of the parabola that satisfies the following conditions: Vertex $(0, 0)$; focus $(3, 0)$.

Solution: Given that, Vertex is $(0, 0)$ and focus is $(3, 0)$.

Here, the vertex of the parabola is $(0, 0)$ and the focus lies on the positive x-axis.

Then, x-axis is the axis of the parabola, while the equation of the parabola is of the form $y^2 = 4ax$. And also the focus is $(3, 0)$.

That is, $a = 3$.

Now, the equation of the parabola is $y^2 = 4 \times 3 \times x$.

Multiply the terms,

$$y^2 = 12x$$

Therefore, the equation of the parabola is $y^2 = 12x$.

Question 10. Find the equation of the parabola that satisfies the following conditions: Vertex $(0,0)$ focus $(-2,0)$.

Solution: Given that, Vertex is $(0,0)$ and focus is $(-2,0)$.

Here, the vertex of the parabola is $(0,0)$ and the focus lies on the negative x-axis.

Then, x axis is the axis of the parabola, while the equation of the parabola is of the form $y^2=-4ax$.

And also observed that the focus is $(-2,0)$.

That is, $a=2$.

Now, the equation of the parabola is $y^2=-4(2)x$

Multiply the terms,

$$y^2=-8x$$

Therefore, the equation of the parabola is $y^2=-8x$.

Question 11. Find the equation of the parabola that satisfies the following conditions: Vertex $(0,0)$ passing through $(2,3)$ and axis is along x-axis.

Solution: Given that, the vertex is $(0,0)$ and the axis of the parabola is the x-axis.

So, the equation of the parabola is either of the form $y^2=4ax$ or $y^2=-4ax$.

Here, the parabola passes through point $(2,3)$ and it lies in the first quadrant.

Thus, the equation of the parabola is of the form $y^2=4ax$, while point $(2,3)$ must satisfy the equation $y^2=4ax$.

Then, $3^2=4a(2) \Rightarrow a=\frac{9}{8}$

Thus, the equation of the parabola is

$$y^2=4\left(\frac{9}{8}\right)x$$

Multiply the numbers and simplify ,

$$y^2=\frac{9}{2}x$$

Multiply both sides by 2,

$$2y^2=9x$$

Question 12. Find the equation of the parabola that satisfies the following conditions: Vertex $(0,0)$, passing through $(5,2)$ and symmetric with respect to y-axis.

Solution: Given that, the vertex is $(0,0)$ and the parabola is symmetric about the y-axis.

So, the equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

Here, the parabola passes through point $(5,2)$ and it lies in the first quadrant.

Thus, the equation of the parabola is of the form $x^2 = 4ay$, while point $(5,2)$ must satisfy the equation $x^2 = 4ay$

$(5)^2 = 4 \times a \times 2$

Determine the square and multiply the terms,

$25 = 8a$

Divide both sides by 8,

$a = \frac{25}{8}$

Now, the equation of the parabola is

$x^2 = 4\left(\frac{25}{8}\right)y$

Multiply the numbers and simplify,

$x^2 = \left(\frac{25}{2}\right)y$

Multiply both sides by 2,

$2x^2 = 25y$

Therefore, the equation of the parabola is $2x^2 = 25y$.

Exercise 11.3

Question 1. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$.

Solution: Given that, the equation of ellipse is $\frac{x^2}{36} + \frac{y^2}{16} = 1$.

It can be observed that the denominator of $\frac{x^2}{36}$ is greater than the denominator of $\frac{y^2}{16}$.

Thus, the major axis is along the x-axis and the minor axis is along the y-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a=6$ and $b=4$.

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$c=\sqrt{36-16}$

Subtract the numbers,

$c=\sqrt{20}$

Simplify the radicand,

$c=2\sqrt{5}$

Therefore, The coordinates of the foci are $(2\sqrt{5},0)$ and $(-2\sqrt{5},0)$.

The coordinates of the vertices are $(6,0)$ and $(-6,0)$.

Length of major axis $=2a=12$

Length of minor axis $=2b=8$

Eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$$e=\frac{2\sqrt{5}}{6}$$

Cancel the numbers,

$$e=\frac{\sqrt{5}}{3}$$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2\times16}{6}=\frac{16}{3}$$

Question 2. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{4}+\frac{y^2}{25}=1$

Solution: Given that, the equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{25}=1$.

This can be written as,

$$\frac{x^2}{2^2}+\frac{y^2}{5^2}=1$$

It can be observed that the denominator of $\frac{y^2}{25}$ is greater than the denominator of $\frac{x^2}{4}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, we get $b=2$ and $a=5$

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$$c=\sqrt{25-4}$$

Subtract the numbers,

$$c=\sqrt{21}$$

Therefore, The coordinates of the foci are $(0,\sqrt{21})$ and $(0,-\sqrt{21})$

The coordinates of the vertices are $(0,5)$ and $(0,-5)$

Length of major axis $=2a=10$

Length of minor axis $=2b=4$

Eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$e=\frac{\sqrt{21}}{5}$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$\frac{2\times 4}{5}=\frac{8}{5}$

Question 3. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Solution: Given that, the equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$.

This can be written as,

$\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

It can be observed that the denominator of $\frac{x^2}{4^2}$ is greater than the denominator of $\frac{y^2}{3^2}$.

Thus, the major axis is along the x-axis and the minor axis is along the y-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a=4$ and $b=3$

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$c=\sqrt{16-9}$

Subtract the numbers,

$c=\sqrt{7}$

Therefore, The coordinates of the foci are $(\sqrt{7},0)$ and $(-\sqrt{7},0)$

The coordinates of the vertices are $(4,0)$ and $(-4,0)$

Length of major axis $=2a=8$

Length of minor axis $=2b=6$

Eccentricity is,

$$e = \frac{c}{a}$$

Substitute the values,

$$e = \frac{c}{a} = \frac{\sqrt{7}}{4}$$

Length of latus rectum $= \frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2 \times 9}{4} = \frac{9}{2}$$

Question 4. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{25} + \frac{y^2}{100} = 1$

Solution: Given that, the equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{100} = 1$.

This can be written as,

$$\frac{x^2}{5^2} + \frac{y^2}{10^2} = 1$$

It can be observed that the denominator of $\frac{y^2}{10^2}$ is greater than the denominator of $\frac{x^2}{5^2}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get $b = 5$ and $a = 10$.

Now, $c = \sqrt{a^2 - b^2}$

Substitute the values,

$$c = \sqrt{100 - 25}$$

Subtract the numbers,

$$c = \sqrt{75}$$

Simplify the radicand,

$c = 5\sqrt{3}$

Therefore, The coordinates of the foci are $\left(0, 5\sqrt{3}\right)$ and $\left(0, -5\sqrt{3}\right)$

The coordinates of the vertices are $(0,10)$ and $(0,-10)$

Length of major axis $= 2a = 20$

Length of minor axis $= 2b = 10$

Eccentricity is,

$$e = \frac{c}{a}$$

Substitute the values,

$$e = \frac{5\sqrt{3}}{10}$$

Cancel the numbers,

$$e = \frac{\sqrt{3}}{2}$$

Length of latus rectum $= \dfrac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2 \times 25}{10} = 5.$$

Question 5. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\dfrac{x^2}{49} + \dfrac{y^2}{36} = 1$

Solution: Given that, the equation of ellipse is $\dfrac{x^2}{49} + \dfrac{y^2}{36} = 1$.

This can be written as,

$$\frac{x^2}{7^2} + \frac{y^2}{6^2} = 1$$

It can be observed that the denominator of $\dfrac{x^2}{7^2}$ is greater than the denominator of $\dfrac{y^2}{6^2}$.

Thus, the major axis is along the x-axis and the minor axis is along the y-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a=7$ and $b=6$

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$$c=\sqrt{49-36}$$

Subtract the numbers,

$$c=\sqrt{13}$$

Therefore, The coordinates of the foci are $\left(\sqrt{13},0\right)$ and $\left(-\sqrt{13},0\right)$

The coordinates of the vertices are $(7,0)$ and $(-7,0)$

Length of major axis $=2a=14$

Length of minor axis $=2b=12$

Eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$$e=\frac{\sqrt{13}}{7}$$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2\times 36}{7}=\frac{72}{7}$$

Question 6. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{100}+\frac{y^2}{400}=1$

Solution: Given that, the equation of ellipse is $\frac{x^2}{100}+\frac{y^2}{400}=1$.

This can be written as,

$$\frac{x^2}{10^2}+\frac{y^2}{20^2}=1$$

It can be observed that the denominator of $\frac{y^2}{20^2}$ is greater than the denominator of $\frac{x^2}{10^2}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing the given equation with the standard equation of ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, we get $b=10$ and $a=20$.

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$$c=\sqrt{400-100}$$

Subtract the numbers,

$$c=\sqrt{300}$$

Simplify the radicand,

$$c=10\sqrt{3}$$

Therefore, The coordinates of the foci are $\left(0,10\sqrt{3}\right)$ and $\left(0,-10\sqrt{3}\right)$

The coordinates of the vertices are $(0,20)$ and $(0,-20)$

Length of major axis $=2a=40$

Length of minor axis $=2b=20$

Eccentricity is,

$$e=\frac{c}{a}$$

Substitute the values,

$$e=\frac{10\sqrt{3}}{20}$$

Cancel the numbers,

$$e=\frac{\sqrt{3}}{2}$$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2\times 100}{20}=10$$

Question 7. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $36x^2+4y^2=144$.

Solution: Given that, the equation is $36x^2+4y^2=144$.

It can be written as

$$\frac{x^2}{4}+\frac{y^2}{36}=1$$

Or $\frac{x^2}{2^2}+\frac{y^2}{6^2}=1\rightarrow(1)$

It can be observed that the denominator of $\frac{y^2}{6^2}$ is greater than the denominator of $\frac{x^2}{2^2}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing equation (1) with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, we get $b=2$ and $a=6$.

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$$c=\sqrt{36-4}$$

Subtract the numbers,

$$c=\sqrt{32}$$

Simplify the radicand,

$$c=4\sqrt{2}$$

Therefore, The coordinates of the foci are $\left(0,4\sqrt{2}\right)$ and $\left(0,-4\sqrt{2}\right)$

The coordinates of the vertices are $(0,6)$ and $(0,-6)$

Length of major axis $=2a=12$

Length of minor axis $=2b=4$

Eccentricity is,

$e=\frac{c}{a}$

Substitute the values,

$e=\frac{4\sqrt{2}}{6}$

Cancel the numbers,

$e=\frac{2\sqrt{2}}{6}$

Length of latus rectum $=\frac{2b^{2}}{a}$

Substitute the values and simplify,

$\frac{2\times4}{6}=\frac{4}{3}$

Question 8. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $16x^{2}+y^{2}=16$.

Solution: Given that, the equation is $16x^{2}+y^{2}=16$.

It can be written as

$\frac{x^{2}}{1}+\frac{y^{2}}{16}=1$

Or, $\frac{x^{2}}{1^{2}}+\frac{y^{2}}{4^{2}}=1\rightarrow(1)$

It can be observed that the denominator of $\frac{y^{2}}{4^{2}}$ is greater than the denominator of $\frac{x^{2}}{1^{2}}$.

Thus, the major axis is along the y-axis and the minor axis is along the x-axis.

When comparing equation (1) with $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$, we get $b=1$ and $a=4$.

Now, $c=\sqrt{a^{2}-b^{2}}$

Substitute the values,

$c=\sqrt{16-1}$

Subtract the numbers,

$$c = \sqrt{15}$$

Therefore, The coordinates of the foci are $\left(0, \sqrt{15}\right)$ and $\left(0, -\sqrt{15}\right)$

The coordinates of the vertices are $(0, 4)$ and $(0, -4)$

Length of major axis $= 2a = 8$

Length of minor axis $= 2b = 2$

Eccentricity is,

$$e = \frac{c}{a}$$

Substitute the values,

$$e = \frac{\sqrt{15}}{4}$$

Length of latus rectum $= \frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2 \times 1}{4} = \frac{1}{2}.$$

Question 9. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse $4x^2 + 9y^2 = 36$.

Solution: Given that, the equation is $4x^2 + 9y^2 = 36$.

It can be written as

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Or, $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1 \to (1)$

It can be observed that the denominator of $\frac{x^2}{3^2}$ is greater than the denominator of $\frac{y^2}{2^2}$.

Thus, the major axis is along the x-axis and the minor axis is along the y-axis.

When comparing the equation (1) with the standard equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a=3$ and $b=2$.

Now, $c=\sqrt{a^2-b^2}$

Substitute the values,

$c=\sqrt{9-4}$

Subtract the numbers,

$c=\sqrt{5}$

Therefore, The coordinates of the foci are $\left(\sqrt{5},0\right)$ and $\left(-\sqrt{5},0\right)$

The coordinates of the vertices are $(3,0)$ and $(-3,0)$

Length of major axis $=2a=6$

Length of minor axis $=2b=4$

Eccentricity is,

$e=\frac{c}{a}$

Substitute the values,

$e=\frac{\sqrt{5}}{3}$

Length of latus rectum $=\frac{2b^2}{a}$

Substitute the values and simplify,

$\frac{2\times 4}{3}=\frac{8}{3}$

Question 10. Find the equation for the ellipse that satisfies the given conditions: Vertices $(\pm 5,0)$, foci $(\pm 4,0)$.

Solution: Given that, the coordinates of vertices are $(\pm 5,0)$ and the coordinates of foci are $(\pm 4,0)$.

Here, the vertices are on the x-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $a=5$ and $c=4$.

It is known that $a^2=b^2+c^2$

That is, $5^2=b^2+4^2$

Determine the squares,

$25=b^2+16$

Subtract 16 from both sides,

$b^2=25-16$

Then, $b=\sqrt{9}=3$

Thus, the equation of the ellipse is $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ or $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Question 11. Find the equation for the ellipse that satisfies the given conditions: Vertices $(0,\pm 13)$, foci $(0,\pm 5)$.

Solution: Given that, the coordinates of vertices are $(0,\pm 13)$ and the coordinates of foci are $(0,\pm 5)$.

Here, the vertices are on the y-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where a is the semi-major axis.

From the given data, $a=13$ and $c=5$.

It is known that $a^2=b^2+c^2$

That is, $13^2=b^2+5^2$

Determine the squares,

$169=b^2+25$

Subtract 25 from both sides,

$b^2=169-25$

Then, $b=\sqrt{144}=12$

Thus, the equation of the ellipse is $\frac{x^2}{12^2}+\frac{y^2}{13^2}=1$ or $\frac{x^2}{144}+\frac{y^2}{169}=1$.

Question 12. Find the equation for the ellipse that satisfies the given conditions: Vertices $(\pm 6, 0)$, foci $(\pm 4, 0)$

Solution: Given that the coordinates of vertices are $(\pm 6, 0)$ and the coordinates of foci are $(\pm 4, 0)$.

Here, the vertices are on the x-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $a=6$ and $c=4$.

It is known that $a^2=b^2+c^2$

That is, $6^2=b^2+4^2$

Determine the squares,

$36=b^2+16$

Subtract 16 from both sides,

$b^2=36-16$

Then, $b=\sqrt{20}$

Thus, the equation of the ellipse is $\frac{x^2}{6^2}+\frac{y^2}{\left(\sqrt{20}\right)^2}=1$ or $\frac{x^2}{36}+\frac{y^2}{20}=1$.

Question 13. Find the equation for the ellipse that satisfies the given conditions: Ends of major axis. $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$.

Solution: Given that, Ends of major axis are $(\pm 3, 0)$ and ends of minor axis are $(0, \pm 2)$.

Here, the major axis is along the x-axis.

Thus, the equation of the ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $a=3$ and $b=2$.

Therefore, the equation of the ellipse is $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ or $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Question 14. Find the equation for the ellipse that satisfies the given conditions: Ends of major axis $\left(0, \pm\sqrt{5}\right)$, ends of minor axis $(\pm 1, 0)$.

Solution: Given that, ends of major axis are $\left(0, \pm\sqrt{5}\right)$ and ends of minor axis are $(\pm 1, 0)$.

Here, the major axis is along the y-axis.

Thus, the equation of the ellipse is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where a is the semi-major axis.

From the given data, $a=\sqrt{5}$ and $b=1$.

Therefore, the equation of the ellipse is $\frac{x^2}{1^2}+\frac{y^2}{(\sqrt{5})^2}=1$ or $\frac{x^2}{1}+\frac{y^2}{5}=1$.

Question 15. Find the equation for the ellipse that satisfies the given conditions: Length of major axis 26, foci $(\pm 5, 0)$

Solution: Given that, Length of major axis $=26$ and foci $=(\pm 5, 0)$.

Here, the foci are on the x-axis.

So, the major axis is along the x-axis.

Therefore, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $2a=26 \Rightarrow a=13$ and $c=5$.

It is known that $a^2=b^2+c^2$

Then, $13^2=b^2+5^2$

Determine the squares,

$169=b^2+25$

Subtract 25 from both sides,

$b^2=169-25$

$b=\sqrt{144}=12$

Therefore, the equation of the ellipse is $\frac{x^2}{13^2}+\frac{y^2}{12^2}=1$ or $\frac{x^2}{169}+\frac{y^2}{144}=1$.

Question 16. Find the equation for the ellipse that satisfies the given conditions: Length of minor axis 16, foci $(0, \pm 6)$.

Solution: Given that, Length of minor axis $=16$ and foci $=(0, \pm 6)$.

Here, the foci are on the y-axis.

So, the major axis is along the y-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where a is the semi-major axis.

From the given data, $2b=16 \Rightarrow b=8$ and $c=6$.

It is known that $a^2=b^2+c^2$.

Then, $a^2=8^2+6^2$

Determine the squares,

$a^2=64+36$

$a^2=100$

$a=\sqrt{100}=10$

Therefore, the equation of the ellipse is $\frac{x^2}{8^2}+\frac{y^2}{10^2}=1$ or $\frac{x^2}{64}+\frac{y^2}{100}=1$.

Question 17. Find the equation for the ellipse that satisfies the given conditions: Foci $(\pm 3,0), a=4$.

Solution: Given that, Foci $(\pm 3,0)$ and $a=4$.

Here, the foci are on the x-axis.

So, the major axis is along the x-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

From the given data, $c=3$ and $a=4$.

It is known that $a^2=b^2+c^2$

Then, $4^2=b^2+3^2$

Determine the squares,

$16=b^2+9$

$b^2=16-9=7$

Therefore, the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{7}=1$.

Question 18. Find the equation for the ellipse that satisfies the given conditions: $b=3, c=4$, centre at the origin; foci on the x axis.

Solution: Given that, $b=3, c=4$, centre at the origin and foci on the x axis.

Here, the foci are on the x-axis.

So, the major axis is along the x-axis.

Thus, the equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

It is known that $a^2=b^2+c^2$.

$a^2=3^2+4^2$

Determine the squares,

$a^2=9+16$

Add the numbers,

$a^2=25$

$a=5$

Therefore, the equation of the ellipse is $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ or $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Question 19. Find the equation for the ellipse that satisfies the given conditions: Centre at $(0,0)$, major axis on the y-axis and passes through the points $(3,2)$ and $(1,6)$.

Solution: Here, the centre is at $(0,0)$ and the major axis is on the y-axis.

So, the equation of the ellipse is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \rightarrow (1)$

Where, a is the semi-major axis.

Given that, the ellipse passes through points $(3,2)$ and $(1,6)$.

Thus, $\frac{9}{b^2}+\frac{4}{a^2}=1 \rightarrow (2)$

$\frac{1}{b^2}+\frac{36}{a^2}=1 \rightarrow (3)$

$\frac{1}{b^2}=1-\frac{36}{a^2}$

Take LCM,

$$\frac{1}{b^2}=\frac{a^2-36}{a^2}\rightarrow(4)$$

Substitute this on (2),

$$9\left(\frac{a^2-36}{a^2}\right)+\frac{4}{a^2}=1$$

Open brackets,

$$\frac{9a^2-324}{a^2}+\frac{4}{a^2}=1$$

Add the terms,

$$\frac{9a^2-320}{a^2}=1$$

Cross multiply,

$$9a^2-320=a^2$$

$$8a^2=320$$

$$a^2=40$$

Substitute the value of a^2 in (4),

$$\frac{1}{b^2}=\frac{40-36}{40}=\frac{4}{40}=\frac{1}{10}$$

Then, $b^2=10$

Therefore, the equation of the ellipse is $\frac{x^2}{10}+\frac{y^2}{40}=1$ or $4x^2+y^2=40$.

Question 20. Find the equation for the ellipse that satisfies the given conditions: Major axis on the x axis and passes through the points $(4,3)$ and $(6,2)$.

Solution: Here, the major axis is on the x-axis.

So, the equation of the ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\rightarrow(1)$

Where, a is the semi-major axis.

Given that, the ellipse passes through points $(4,3)$ and $(6,2)$.

Thus, $\frac{16}{a^2}+\frac{9}{b^2}=1\rightarrow(2)$

$$\frac{36}{a^2}+\frac{4}{b^2}=1 \rightarrow (3)$$

Subtract (3) from (2),

$$\frac{-20}{a^2}+\frac{5}{b^2}=0$$

$$\Rightarrow 5a^2=20b^2$$

$$\Rightarrow a^2=4b^2$$

Putting the value of a^2 in (2)

$$\frac{16}{4b^2}+\frac{9}{b^2}=1$$

$\therefore b^2=13$ and $a^2=4b^2=4\times 13=52$

Therefore, the equation of the ellipse is $\frac{x^2}{52}+\frac{y^2}{13}=1$ or $x^2+4y^2=52$.

Exercise 11.4

Question 1. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Solution: Given that, the equation is $\frac{x^2}{16}-\frac{y^2}{9}=1$.

This can be written as,

$$\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$$

When comparing this equation with the standard equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a=4$ and $b=3$.

It is known that $a^2+b^2=c^2$.

Then, $c^2=4^2+3^2$

Determine the squares and add the numbers,

$$c^2=25$$

Thus, $c=5$.

Therefore,

The coordinates of the foci are $(\pm 5, 0)$.

The coordinates of the vertices are $(\pm 4, 0)$.

Eccentricity is,

$$e = \frac{c}{a}$$

Substitute the values,

$$e = \frac{5}{4}$$

Length of latus rectum is,

$$\frac{2b^2}{a}$$

Substitute the values and simplify,

$$\frac{2 \times 9}{4} = \frac{9}{2}$$

Question 2. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $\frac{y^2}{9} - \frac{x^2}{27} = 1$.

Solution: Given that, the equation is $\frac{y^2}{9} - \frac{x^2}{27} = 1$.

This can be written as,

$$\frac{y^2}{3^2} - \frac{x^2}{(\sqrt{27})^2} = 1$$

When comparing this equation with the standard equation of hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2}$, we get $a = 3$ and $b = \sqrt{27}$.

It is known that $a^2 + b^2 = c^2$

Now, $c^2 = 3^2 + \left(\sqrt{27}\right)^2$

Determine the squares and add the numbers,

$$c^2 = 36$$

Then, $c = 6$

Therefore,

The coordinates of the foci are $(0, \pm 6)$.

The coordinates of the vertices are $(0, \pm 3)$.

Eccentricity, $e = \frac{c}{a}$

Substitute the values,

$e = \frac{6}{3}$

Divide the numbers,

$e = 2$

Length of latus rectum $= \frac{2b^2}{a}$

Substitute the values and simplify,

$\frac{2 \times 27}{3} = 18$

Question 3. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $9y^2 - 4x^2 = 36$.

Solution: Given that, the given equation is $9y^2 - 4x^2 = 36$.

This can be written as

$\frac{y^2}{4} - \frac{x^2}{9} = 1$

Or, $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$

When comparing the equation $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$ with the standard equation of hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get $a = 2$ and $b = 3$.

It is known that $a^2 + b^2 = c^2$

Now, $c^2 = 4 + 9$

Add the numbers,

$c^2 = 13$

Then, $c = \sqrt{13}$

Therefore,

The coordinates of the foci are $(0, \pm\sqrt{13})$

The coordinates of the vertices are $(0, \pm 2)$

Eccentricity, $e = \dfrac{c}{a}$

Substitute the values,

$$e = \frac{\sqrt{13}}{2}$$

Length of latus rectum $= \dfrac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2 \times 9}{2} = 9$$

Question 4. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$

Solution: Given that, the equation is $16x^2 - 9y^2 = 576$.

It can be written as,

Or $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$

Or $\dfrac{x^2}{6^2} - \dfrac{y^2}{8^2} = 1$

When comparing the equation $\dfrac{x^2}{6^2} - \dfrac{y^2}{8^2} = 1$ with the standard equation of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get $a = 6$ and $b = 8$.

It is known that $a^2 + b^2 = c^2$

Now, $c^2 = 36 + 64$

Add the numbers,

$c^2 = 100$

Then, $c = 10$

Therefore,

The coordinates of the foci are $(\pm 10, 0)$

The coordinates of the vertices are $(\pm 6, 0)$

Eccentricity, $e = \frac{c}{a}$

Substitute the values,

$$e = \frac{10}{6}$$

Simplify,

$$e = \frac{5}{3}$$

Length of latus rectum $= \frac{2b^2}{a}$

Substitute the values and simplify,

$$\frac{2 \times 64}{6} = \frac{64}{3}$$

Question 5. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $5y^2 - 9x^2 = 36$.

Solution: Given that, the equation is $5y^2 - 9x^2 = 36$.

It can be written as,

$$\frac{y^2}{\left(\frac{36}{5}\right)} - \frac{x^2}{4} = 1$$

Or $\frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)^2} - \frac{x^2}{2^2} = 1$

When comparing the equation this with the standard equation of hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get

$a = \frac{6}{\sqrt{5}}$ and $b = 2$

It is known that $a^2+b^2=c^2$.

Then, $c^2=\frac{36}{5}+4=\frac{56}{5}$

$\Rightarrow c=\sqrt{\frac{56}{5}}=\frac{2\sqrt{14}}{\sqrt{5}}$

Thus, the coordinates of the foci are $\left(0,\pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$

The coordinates of the vertices are $\left(0,\pm\frac{6}{\sqrt{5}}\right)$.

Eccentricity, $e=\frac{c}{a}$

Substitute the values and simplify,

$$e=\frac{\left(\frac{2\sqrt{14}}{\sqrt{5}}\right)}{\left(\frac{6}{\sqrt{5}}\right)}=\frac{\sqrt{14}}{3}$$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 4}{\left(\frac{6}{\sqrt{5}}\right)}=\frac{4\sqrt{5}}{3}$

Question 6. Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $49y^2-16x^2=784$

Solution: Given that, the equation is $49y^2-16x^2=784$.

It can be written as

$49y^2-16x^2=784$

Or, $\frac{y^2}{16}-\frac{x^2}{49}=1$

Or, $\frac{y^2}{4^2}-\frac{x^2}{7^2}=1$

When comparing this equation with the standard equation of hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we get $a=4$ and $b=7$.

It is known that $a^2+b^2=c^2$.

Then, $c^2=16+49=65$

$\Rightarrow c=\sqrt{65}$

Therefore,

The coordinates of the foci are $(0,\pm\sqrt{65})$.

The coordinates of the vertices are $(0,\pm 4)$.

Eccentricity, $e=\frac{c}{a}=\frac{\sqrt{65}}{4}$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 49}{4}=\frac{49}{2}$

Question 7. Find the equation of the hyperbola satisfying the give conditions: Vertices $(\pm 2,0)$, foci $(\pm 3,0)$

Solution: Given that, Vertices $(\pm 2,0)$ and foci $(\pm 3,0)$

Here, the vertices are on the x-axis.

Then, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Given the vertices are $(\pm 2,0)$.

That is, $a=2$

Since the foci are $(\pm 3,0)$,

$c=3$

It is known that $a^2+b^2=c^2$.

$2^2+b^2=3^2$

$b^2=9-4=5$

Therefore, the equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{5}=1$.

Question 8. Find the equation of the hyperbola satisfying the give conditions: Vertices $(0,\pm 5)$, foci $(0,\pm 8)$

Solution: Given that, Vertices $(0,\pm 5)$ and foci $(0,\pm 8)$.

Here, the vertices are on the y-axis.

Then, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Given the vertices are $(0,\pm 5)$

That is, $a=5$

Since the foci are $(0,\pm 8)$,

$c=8$

It is known that $a^2+b^2=c^2$

Then, $5^2+b^2=8^2$

$b^2=64-25=39$

Therefore, the equation of the hyperbola is $\frac{y^2}{25}-\frac{x^2}{39}=1$.

Question 9. Find the equation of the hyperbola satisfying the give conditions: Vertices $(0,\pm 3)$, foci $(0,\pm 5)$.

Solution: Given that, Vertices $(0,\pm 3)$ and foci $(0,\pm 5)$.

Here, the vertices are on the y-axis.

Then, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Here, the vertices are $(0,\pm 3)$.

That is, $a=3$

Since the foci are $(0,\pm 5)$,

$c=5$

It is known that $a^2+b^2=c^2$.

$3^2+b^2=5^2$

$b^2=25-9=16$

Therefore, the equation of the hyperbola is $\frac{y^2}{9}-\frac{x^2}{16}=1$.

Question 10. Find the equation of the hyperbola satisfying the give conditions: Foci $(\pm 5, 0)$, the transverse axis is of length 8.

Solution: Given that, Foci $(\pm 5, 0)$ and the transverse axis is of length 8 .

Here, the foci are on the x-axis.

Then, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the foci are $(\pm 5, 0)$,

$c = 5$

Since the length of the transverse axis is 8,

$2a = 8 \Rightarrow a = 4$

It is known that $a^2 + b^2 = c^2$

$4^2 + b^2 = 5^2$

$b^2 = 25 - 16 = 9$

Therefore, the equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Question 11. Find the equation of the hyperbola satisfying the give conditions: Foci $(0, \pm 13)$, the conjugate axis is of length 24.

Solution: Given that, Foci $(0, \pm 13)$ and the conjugate axis is of length 24.

Here, the foci are on the y-axis.

Then, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the foci are $(0, \pm 13)$,

$c = 13$

Since the length of the conjugate axis is 24,

$2b = 24 \Rightarrow b = 12$

It is known that $a^2 + b^2 = c^2$

$a^2 + 12^2 = 13^2$

$a^2 = 169 - 144 = 25$

Therefore, the equation of the hyperbola is $\frac{y^2}{25}-\frac{x^2}{144}=1$.

Question 12. Find the equation of the hyperbola satisfying the give conditions: Foci $(\pm 3\sqrt{5},0)$, the latus rectum is of length 8.

Solution: Given that, Foci $\left(\pm 3\sqrt{5},0\right)$ and the latus rectum is of length 8.

Here, the foci are on the x-axis.

Then, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Since the foci are $\left(\pm 3\sqrt{5},0\right)$,

$c=\pm 3\sqrt{5}$

Length of latus rectum $=8$

That is, $\frac{2b^2}{a}=8$

$b^2=4a$

It is known that $a^2+b^2=c^2$.

$a^2+4a=45$

Subtract 45 from both sides,

$a^2+4a-45=0$

Rewrite the equation,

$a^2+9a-5a-45=0$

Factorize the expression,

$(a+9)(a-5)=0$

By zero product property,

$a=-9,5$

Since a is non-negative,

$a=5$.

Now, $b^2=4a=4\times 5=20$

Therefore, the equation of the hyperbola is $\frac{x^2}{25}-\frac{y^2}{20}=1$.

Question 13. Find the equation of the hyperbola satisfying the give conditions: Foci $(\pm 4,0)$, the latus rectum is of length 12.

Solution: Given that, Foci $(\pm 4,0)$ and the latus rectum is of length 12.

Here, the foci are on the x-axis.

Then, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Since the foci are $(\pm 4,0)$,

$c=4$

Length of latus rectum $=12$

That is, $\frac{2b^2}{a}=12$

$b^2=6a$

It is known that $a^2+b^2=c^2$.

$a^2+6a=16$

Rewrite the equation,

$a^2+8a-2a-16=0$

Factorize the expression,

$(a+8)(a-2)=0$

By zero product property,

$a=-8,2$

Since a is non-negative,

$a=2$.

$b^2=6a=6\times 2=12$

Therefore, the equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$.

Question 14. Find the equation of the hyperbola satisfying the give conditions: Vertices $(\pm 7,0)$

$e=\frac{4}{3}$.

Solution: Given that, the Vertices $(\pm 7,0)$ and $e=\frac{4}{3}$.

Here, the vertices are on the x-axis.

Then, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Since the vertices are $(\pm 7,0)$,

$a=7$

It is given that $e=\frac{4}{3}$

$\frac{c}{a}=\frac{4}{3}\quad\left[e=\frac{c}{a}\right]$

Substitute the value,

$\frac{c}{7}=\frac{4}{3}$

Multiply both sides by 7,

$c=\frac{28}{3}$

It is known that $a^2+b^2=c^2$.

Now, $7^2+b^2=\left(\frac{28}{3}\right)^2$

Determine the squares and subtract 7^2 from both sides,

$b^2=\frac{784}{9}-49$

Take LCM,

$b^2=\frac{784-441}{9}=\frac{343}{9}$

Therefore, the equation of the hyperbola is $\frac{x^2}{49}-\frac{9y^2}{343}=1$.

Question 15. Find the equation of the hyperbola satisfying the give conditions: Foci $(0, \pm\sqrt{10})$ passing through $(2,3)$.

Solution: Given that, Foci $\left(0, \pm\sqrt{10}\right)$ and passing through $(2,3)$

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Since the foci are $\left(0, \pm\sqrt{10}\right)$,

$c = \sqrt{10}$

It is known that $a^2 + b^2 = c^2$

Substitute the values,

$a^2 + b^2 = 10$

$b^2 = 10 - a^2 \rightarrow (1)$

Since the hyperbola passes through point $(2,3)$,

$$\frac{9}{a^2} - \frac{4}{b^2} = 1 \rightarrow (2)$$

From equations (1) and (2), we get

$$\frac{9}{a^2} - \frac{4}{\left(10 - a^2\right)} = 1$$

Take LCM,

$9\left(10 - a^2\right) - 4a^2 = a^2\left(10 - a^2\right)$

Open brackets,

$90 - 9a^2 - 4a^2 = 10a^2 - a^4$

Combine the like terms,

$a^4 - 23a^2 + 90 = 0$

Rewrite the equation,

$a^4 - 18a^2 - 5a^2 + 90 = 0$

$a^2\left(a^2 - 18\right) - 5\left(a^2 - 18\right) = 0$

$\left(a^2-18\right)\left(a^2-5\right)=0$

By zero product property,

$a^2=18$ or 5

In hyperbola $c>a$,

$c^2>a^2$

$a^2=5$

$b^2=10-a^2=10-5=5$

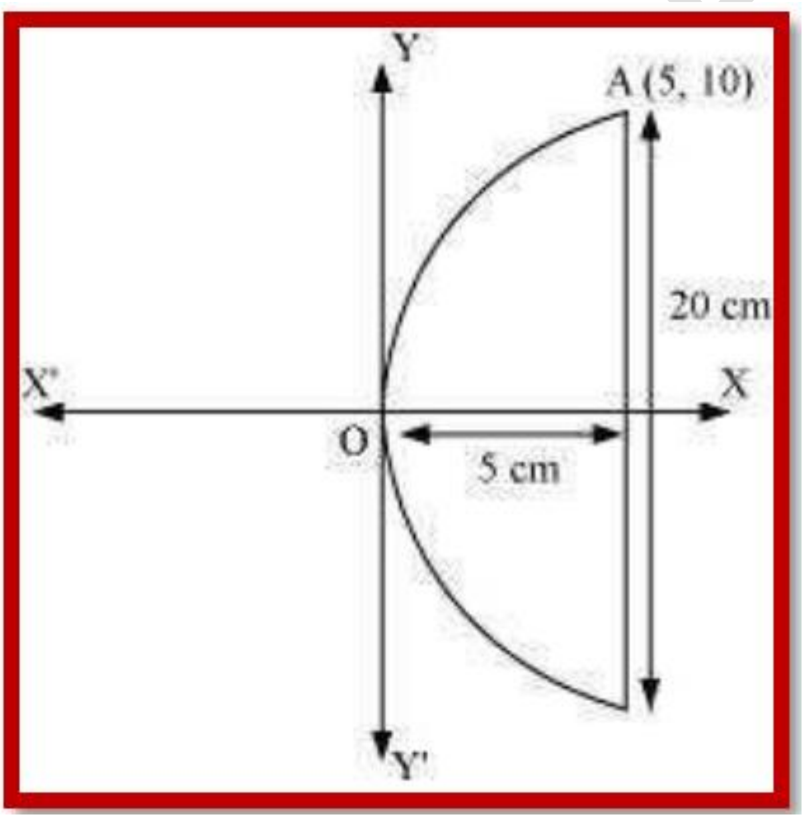
Therefore, the equation of the hyperbola is $\frac{y^2}{5}-\frac{x^2}{5}=1$.

Miscellaneous Exercise

Question 1. If a parabolic reflector is 20cm in diameter and 5cm deep, find the focus.

Solution: The origin of the coordinate plane is taken at the vertex of the parabolic reflector in such a way that the axis of the reflector is along the positive x-axis.

This can be represented as,



The equation of the parabola is of the form $y^2=4ax$ (as it is opening to the right).

Since the parabola passes through point $A(10,5)$,

$10^2 = 4a(5)$

Determine the square and multiply the numbers,

$100 = 20a$

Divide both sides by a,

$$a = \frac{100}{20} = 5$$

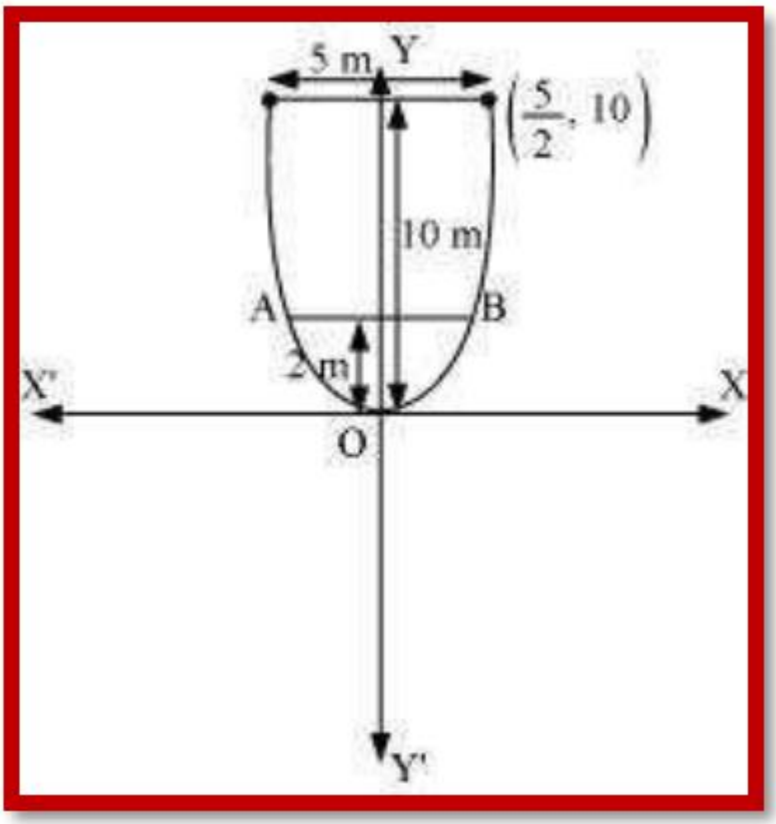
Hence, the focus of the parabola is $(a,0) = (5,0)$, which is the mid-point of the diameter.

Therefore, the focus of the reflector is at the mid-point of the diameter.

Question 2. An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

Solution: The origin of the coordinate plane is taken at the vertex of the arch in such a way that its vertical axis is along the positive y-axis.

This can be represented as,



Here, the equation of the parabola is of the form $x^2 = 4ay$ (as it is opening upwards).

It can be clearly observed that the parabola passes through point $\left(\frac{5}{2},10\right)$.

$$\left(\frac{5}{2}\right)^2 = 4a(10) \Rightarrow a = \frac{25}{4\times4\times10} = \frac{5}{32}$$

Therefore, the arch is in the form of a parabola whose equation is $x^2 = \frac{5}{8}y$

When $y = 2\,\text{m}$,

$$x^2 = \frac{5}{8}\times2$$

Cancel the numbers,

$$\Rightarrow x^2 = \frac{5}{4}$$

$$\Rightarrow x = \sqrt{\frac{5}{4}}\,\text{m}$$

Hence, $\text{AB} = 2\times\sqrt{\frac{5}{4}}\,\text{m} \approx 2\times1.118\,\text{m} \approx 2.23\,\text{m}.$

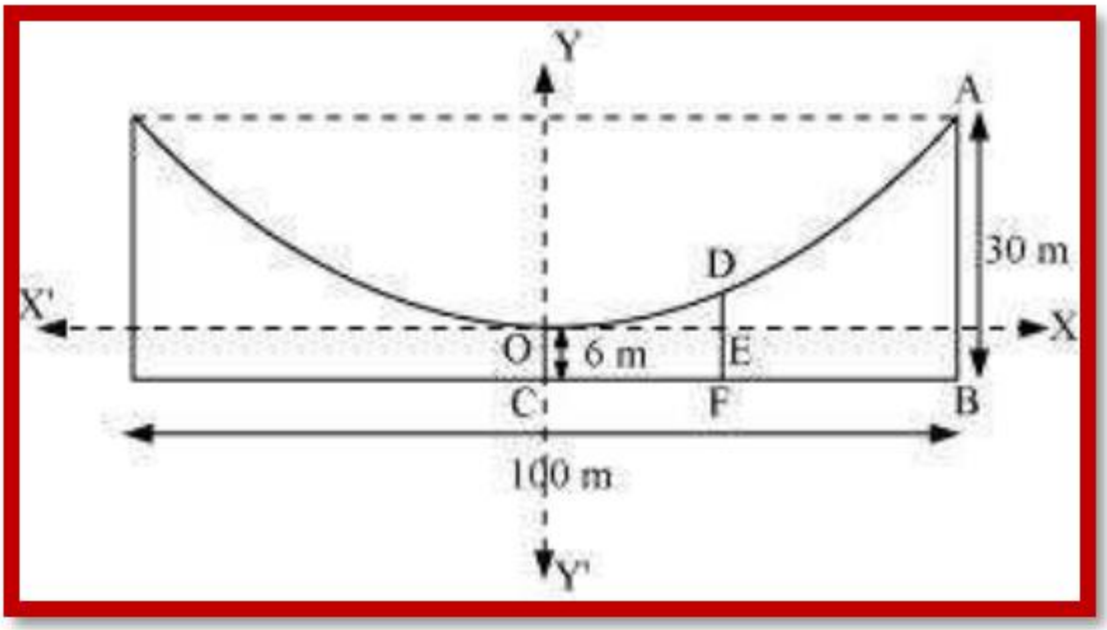
Therefore, when the arch is $2\,\text{m}$ from the vertex of the parabola, its width is approximately $2.23\,\text{m}$.

Question 3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and $100\,\text{m}$ long is supported by vertical wires attached to the cable, the longest wire being $30\,\text{m}$ and the shortest being $6\,\text{m}$. Find the length of a supporting wire attached to the roadway $18\,\text{m}$ from the middle.

Solution: The vertex is at the lowest point of the cable.

The origin of the coordinate plane is taken as the vertex of the parabola, while its vertical axis is taken along the positive y-axis.

This can be represented as,



Here, AB and OC are the longest and the shortest wires, respectively, attached to the cable.

DF is the supporting wire attached to the roadway, $18\,\text{m}$ from the middle.

Here, $AB = 30\,\text{m}, OC = 6\,\text{m}$, and $BC = \dfrac{100}{2} = 50\,\text{m}$

The equation of the parabola is of the form $x^2 = 4ay$ (as it is opening upwards).

The coordinates of point A are $(50, 30-6) = (50, 24)$.

Since $A(50, 24)$ is a point on the parabola,

$(50)^2 = 4a(24)$

Simplify the equation,

$$\Rightarrow a = \frac{50\times 50}{4\times 24} = \frac{625}{24}$$

Equation of the parabola is,

$$x^2 = 4\times\frac{625}{24}\times y \text{ or } 6x^2 = 625y$$

The x-coordinate of point D is 18.

So, at $x = 18$,

$6(18)^2 = 625y$

$$y = \frac{6\times 18\times 18}{625}$$

$\Rightarrow y \approx 3.11$

That is, $DE = 3.11m$

$DF = DE + EF = 3.11m + 6m = 9.11m$

Thus, the length of the supporting wire attached to the roadway 18m from the middle is approximately 9.11m.

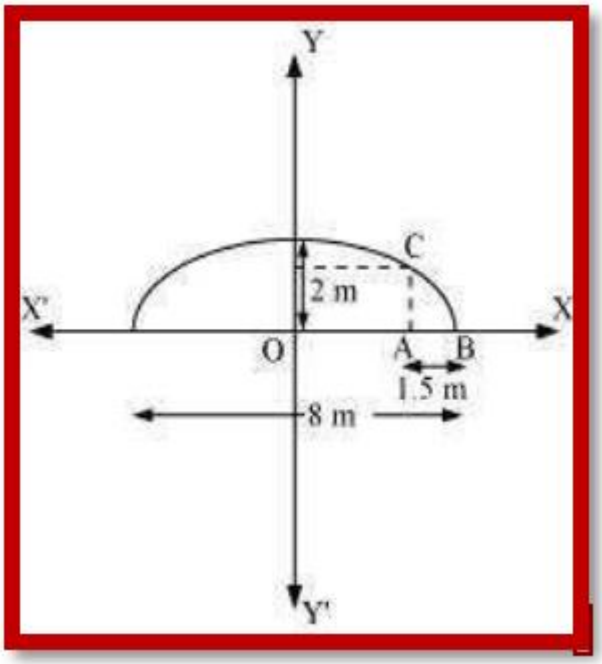
Question 4. An arch is in the form of a semi-ellipse. It is 8m wide and 2m high at the centre. Find the height of the arch at a point 1.5m from one end.

Solution: Here, the height and width of the arc from the centre is 2m and 8m respectively.

It is clear that the length of the major axis is 8m and the length of the semi-minor axis is 2m.

The origin of the coordinate plane is taken as the centre of the ellipse, while the major axis is taken along the x-axis.

Thus, the semi-ellipse can be represented as,



The equation of the semi-ellipse will be of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, y \geq 0$ where a is the semi-major axis.

Then, $2a = 8 \Rightarrow a = 4$

$b = 2$

Thus, the equation of the semi-ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1, y \geq 0 \rightarrow (1)$

Let A be a point on the major axis such that $AB = 1.5\text{m}$.

Draw $AC \perp OB$.

Now, $\text{OA} = (4 - 1.5)\text{m} = 2.5\text{m}$

The x-coordinate of point C is 2.5.

When substituting the value of x with 2.5 in equation (1), we obtain

$$\frac{(2.5)^2}{16} + \frac{y^2}{4} = 1$$

Determine the squares,

$$\frac{6.25}{16} + \frac{y^2}{4} = 1$$

Rewrite the equation,

$$y^2 = 4\left(1 - \frac{6.25}{16}\right)$$

Subtract the numbers,

$$y^2 = 4\left(\frac{9.75}{16}\right)$$

Divide and multiply the numbers,

$$y^2 = 2.4375$$

$$y \approx 1.56$$

That is, $\text{AC} = 1.56\text{m}$

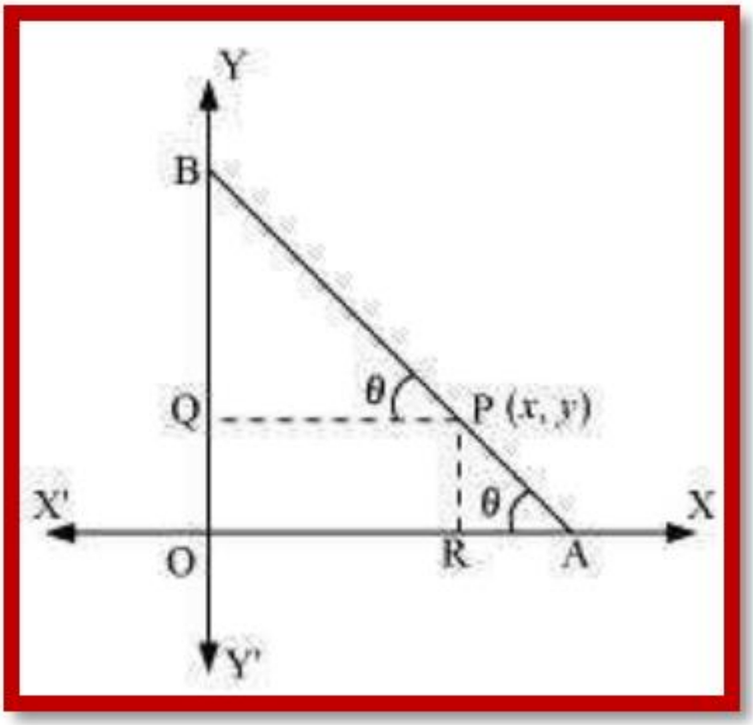
Therefore, the height of the arch at a point 1.5m from one end is approximately 1.56m.

Question 5. A rod of length 12cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3cm from the end in contact with the x-axis.

Solution: Let AB be the rod making an angle θ with OX and P(x,y) be the point on it such that AP=3cm.

Then, $\text{PB=AB-AP=(12-3)cm=9 cm[AB=12 cm]}$

From P, draw $\text{PQ} \perp \text{OY}$ and $\text{PR} \perp \text{OX}$.



In ΔPBQ,

$$\cos\theta = \frac{PQ}{PB} = \frac{x}{9}$$

In ΔPRA,

$$\sin\theta = \frac{PR}{PA} = \frac{y}{3}$$

We know that $\sin^2\theta + \cos^2\theta = 1$

Then, $\left(\frac{y}{3}\right)^2 + \left(\frac{x}{9}\right)^2 = 1$

Or, $\frac{x^2}{81} + \frac{y^2}{9} = 1$

Therefore, the equation of the locus of point P on the rod is $\frac{x^2}{81} + \frac{y^2}{9} = 1$.

Question 6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

Solution: Given that, the equation of parabola is $x^2 = 12y$.

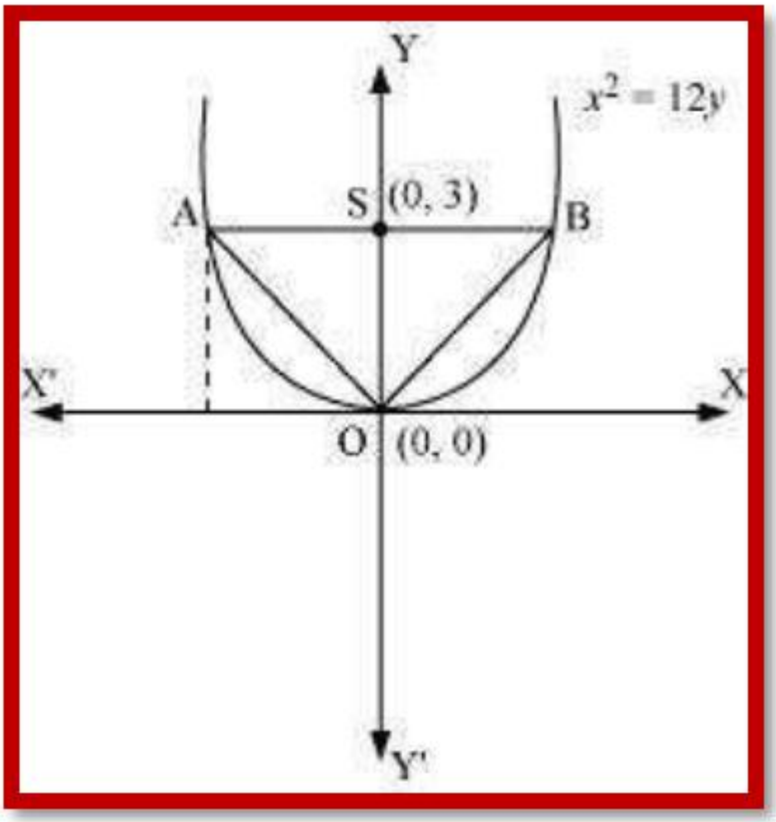
When comparing this equation with $x^2 = 4ay$, we get

$4a = 12 \Rightarrow a = 3$

Thus, the coordinates of foci are $S(0,a) = S(0,3)$

Let AB be the latus rectum of the given parabola.

The given parabola can be roughly drawn as,



At $y = 3$,

$x^2 = 12(3)$

Multiply the numbers and simplify,

$x^2 = 36$

$x = \pm 6$

Now, The coordinates of A are $(-6,3)$, while the coordinates of B are $(6,3)$.

Thus, the vertices of ΔOAB are $O(0,0), A(-6,3)$, and $B(6,3)$.

Then, Area of $\Delta OAB = \frac{1}{2}|0(3-3)+(-6)(3-0)+6(0-3)|$ unit ^{2}

$$\frac{1}{2}|(-6)(3)+6(-3)| \text{ unit }^2$$

Multiply the numbers,

$$\frac{1}{2}|-18-18| \text{ unit }^2$$

Subtract the numbers,

$$\frac{1}{2}|-36| \text{ unit }^2$$

$$\frac{1}{2}\times 36 \text{ unit }^2$$

Multiply the numbers,

18 unit ^2

Therefore, the required area of the triangle is 18 unit ^2.

Question 7. A man running a racecourse notes that the sum of the distances from the two flag posts form him is always $10\mathrm{m}$ and the distance between the flag posts is $8\mathrm{m}$. Find the equation of the posts traced by the man.

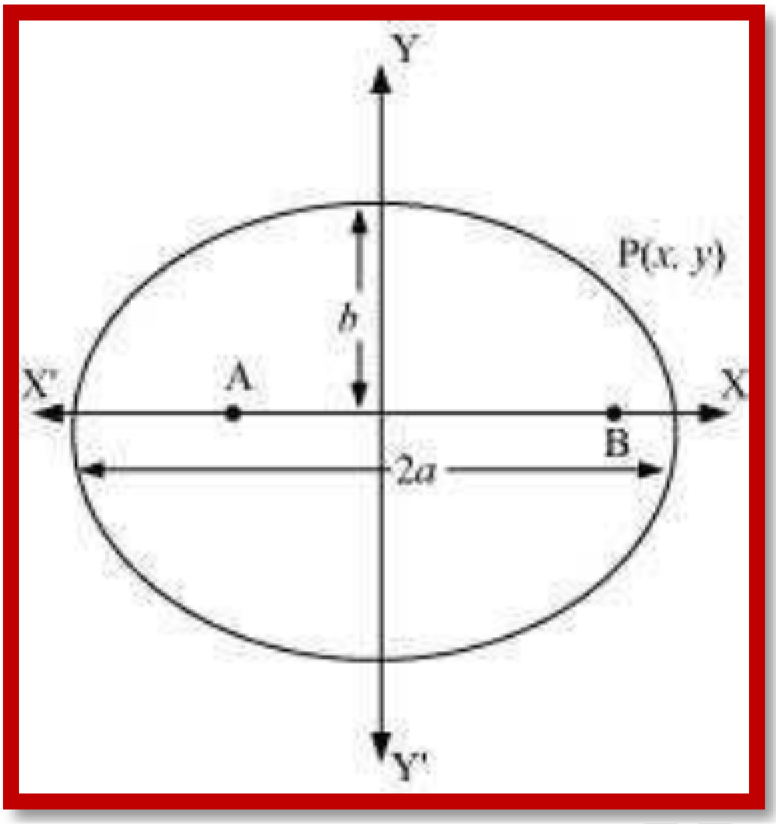
Solution: Let A and B be the positions of the two flag posts and $P(x, y)$ be the position of the man.

Then, $PA + PB = 10$

We know that if a point moves in a plane in such a way that the sum of its distances from two fixed points is constant, then the path is an ellipse and this constant value is equal to the length of the major axis of the ellipse.

Thus, the path described by the man is an ellipse where the length of the major axis is $10\mathrm{m}$, while points A and B are the foci.

Taking the origin of the coordinate plane as the centre of the ellipse, while taking the major axis along the x-axis, the ellipse can be represented as,



The equation of the ellipse will be of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a is the semi-major axis.

Then, $2a=10 \Rightarrow a=5$

Distance between the foci $(2c)=8 \Rightarrow c=4$

On using the relation $c=\sqrt{a^2-b^2}$

$4=\sqrt{25-b^2}$

Square both sides,

$16=25-b^2$

$b^2=25-16$

Subtract the numbers,

$b^2=9$

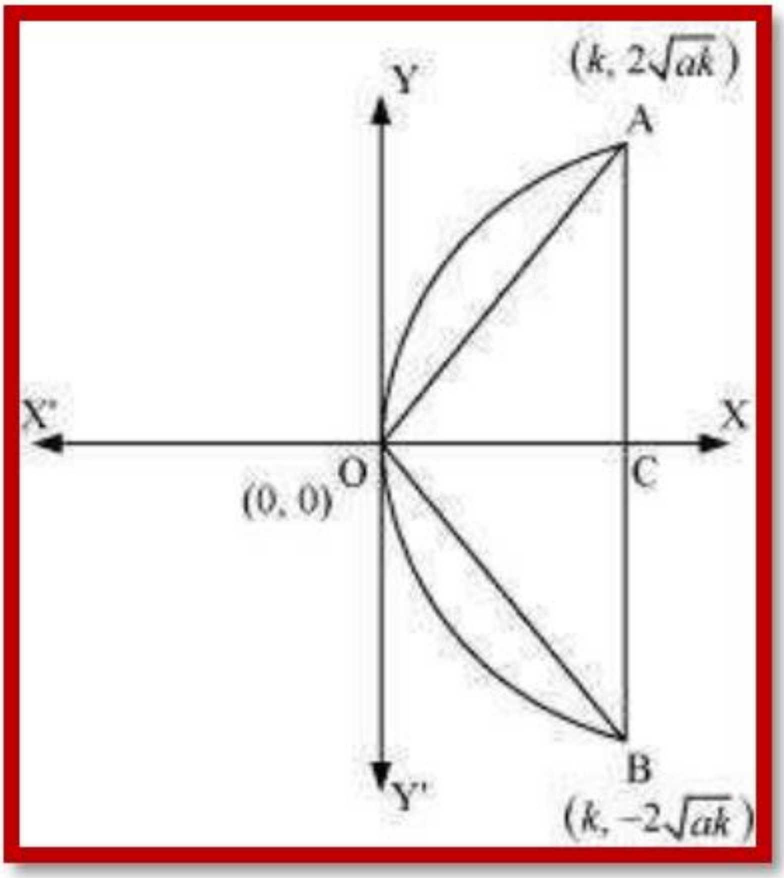
$b=3$

Therefore, the equation of the path traced by the man is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Question 8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution: Let OAB be the equilateral triangle inscribed in parabola $y^2 = 4ax$.

Let AB intersect the x-axis at point C.



Let $\mathrm{OC} = k$

From the equation of the given parabola, we have $y^2 = 4ak \Rightarrow y = \pm 2\sqrt{ak}$.

Then, the respective coordinates of points A and B are $(k, 2\sqrt{ak})$, and $(k, -2\sqrt{ak})$

$\mathrm{AB} = \mathrm{CA} + \mathrm{CB}$

Substitute the values,

$2\sqrt{ak} + 2\sqrt{ak}$

Combine like terms,

$4\sqrt{ak}$

Since OAB is an equilateral triangle, $\mathrm{OA}^2 = \mathrm{AB}^2$.

$$k^2+\left(2\sqrt{ak}\right)^2=\left(4\sqrt{ak}\right)^2$$

Determine the squares,

$$k^2+4ak=16ak$$

Subtract $4ak$ from both sides,

$$k^2=12ak$$

$$k=12a$$

Now, $\mathrm{AB}=4\sqrt{ak}$

Substitute the values,

$$4\sqrt{a\times 12a}$$

Multiply the numbers,

$$4\sqrt{12a^2}=8\sqrt{3}a$$

Therefore, the side of the equilateral triangle inscribed in parabola $y^2=4$ axis $8\sqrt{3}a$.