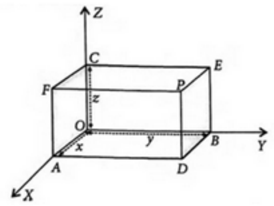
Chapter 12: Introduction to Three Dimensional Geometry

Example 1

In Fig 12.3, if P is $(2,4,5)$, find the coordinates of F.

Solution

From figure,



At the point F the distance measured at OY is zero.

Hence, the coordinates of F will be $(2,0,5)$.

Example 2

Find the octant in which the points $(-3,1,2)$ and $(-3,1,-2)$ lie.

Solution

From the Table 12.1,

At the point $(-3,1,2)$

So, x says negative, y says positive and z says positive

So, it lies in second octant

At the point $(-3,1,-2)$

So, x says negative, y says positive, and z says negative

So, it lies in sixth octant.

EXERCISE 12.1

1. A point is on the x-axis. What are its y-coordinate and z-coordinates?

Solution

If a point is on the x axis its y and z coordinates are 0

If a point in three-dimensional geometry is on the x-axis, then the y and z-coordinates of the points are zero.

Coordinates of any point in x-axis is $(x, 0, 0)$

2. A point is in the XZ-plane. What can you say about its y-coordinate?

Solution

At any point on $\mathrm{XZ}-$plane is $(x, 0, z)$.

Hence, y-coordinate of the point is zero.

3. Name the octants in which the following points lie:

$(1,2,3),(4,-2,3),(4,-2,-5),(4,2,-5),(-4,2,-5),(-4,2,5)$

$(-3,-1,6)(-2,-4,-7)$

Solution

Table which represents octants

Octants	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

(i) For the point $(1,2,3)$

$x=1>0$ is measured along OX

$y=2>0$ is measured along OY

$z=3>0$ is measured along OZ

Hence the point $(1,2,3)$ lies in the octant XOYZ,

That is in octant I.

(ii) For the point $(4,-2,3)$

$x=4>0$ is measured along OX

$y=-2<0$ is measured along OY

$z=3>0$ is measured along OZ

Hence the point $(4,-2,3)$ lies in the octant XOYZ,

That is in octant IV.

(iii) For the point $(4,-2,-5)$

$x=4>0$ is measured along OX

$y = -2 < 0$ is measured along OY'

$z = -5 < 0$ is measured along OZ'

Hence the point $(4, -2, -5)$ lies in the octant $XOYZ'$,

That is in octant VIII.

(iv) For the point $(4, 2, -5)$

$x = 4 > 0$ is measured along OX

$y = 2 > 0$ is measured along OY

$z = -5 < 0$ is measured along OZ

Hence the point $(4, 2, -5)$ lies in the octant $XOYZ$,

That is in octant V.

(v) For the point $(-4, 2, -5)$

$x = -4 < 0$ is measured along OX

$y = 2 > 0$ is measured along OY

$z = -5 < 0$ is measured along OZ

Hence the point $(-4, 2, -5)$ lies in the octant $XOYZ$,

That is in octant VI.

(vi) For the point $(-4, 2, 5)$

$x = -4 < 0$ is measured along OX

$y = 2 > 0$ is measured along OY

$z = 5 > 0$ is measured along OZ

Hence the point $(-4, 2, 5)$ lies in the octant $XOYZ$,

That is in octant II.

(vii) For the point $(-3, -1, 6)$

$x = -3 < 0$ is measured along OX'

$y = -1 < 0$ is measured along OY'

$z = 6 > 0$ is measured along OZ

Hence the point $(-3, -1, 6)$ lies in the octant $X'OY'Z$,

That is, in octant III.

(viii) For the point $(2,-4,-7)$

$x = 2 > 0$ is measured along OX

$y = -4 < 0$ is measured along OY'

$z = -7 < 0$ is measured along OZ'

Hence the point $(2,-4,-7)$ lies in the octant $XOY'Z'$,

That is in octant VIII.

4. Fill in the blanks:

(i) The x-axis and y-axis taken together determine a plane known as

(ii) The coordinates of points in the XY-plane are of the form ………………

(iii) Coordinate planes divide the space into octants.

Solution

(i) The x-axis and y-axis taken together determine a plane known as $\underline{xy}$ - plane.

(ii) The coordinates of points in the XY- plane are of the form $(x, y, 0)$.

(iii) Coordinate planes divide the space into eight octants.

Example 3

Find the distance between the points $P(1,-3,4)$ **and** $Q(-4,1,2)$**.**

Solution

The distance PQ between the points $P(1,-3,4)$ and $Q(-4,1,2)$ is

The coordinates of points

$x_1 = 1, y_1 = -3, z_1 = 4$

$x_2 = -4, y_2 = 1, z_2 = 2$

Distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Substitute the values

$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2}$

$= \sqrt{25+16+4}$

$= \sqrt{45}$

$= 3\sqrt{5}$ units

The distance between the points is $3\sqrt{5}$ units.

Example 4

Show that the points $P(-2,3,5), Q(1,2,3)$ and $R(7,0,-1)$ are collinear.

Solution

The points are said to be collinear if they lie on a line.

Let first calculate distance between the 3 points

It is PQ,QR and PR

Calculating PQ

$P(-2,3,5)$ and $Q(1,2,3)$

$R(7,0,-1)$

Now,

$$PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$$
$$= \sqrt{9+1+4}$$
$$= \sqrt{14}$$

Same as distance of QR

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$$
$$= \sqrt{36+4+16}$$
$$= \sqrt{56}$$
$$= 2\sqrt{14}$$

And

Distance of PR

$$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$$
$$= \sqrt{81+9+36}$$
$$= \sqrt{126}$$
$$= 3\sqrt{14}$$

So, $PQ + QR = PR$.

Therefore, P, Q and R are collinear.

Example 5

Are the points A $(3,6,9)$, B $(10,20,30)$ and C $(25,-41,5)$, the vertices of a right angled triangle?

Solution

Let's first calculate distances AB, BC and AC & then apply Pythagoras theorem

To check whether it is right triangle

Calculating AB

$A(3,6,9)$
$B(10,20,30)$

C $(25,-41,5)$

Distance formula $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

The coordinates of points A and B

Here, $x_1=3, y_1=6, z_1=9$

$x_2=10, y_2=20, z_2=30$

Substitute the values in distance formula

$AB=\sqrt{(10-3)^2+(20-6)^2+(30-9)^2}$

$=\sqrt{(7)^2+(14)^2+(21)^2}$

$=\sqrt{49+196+441}$

$=\sqrt{686}$

$AB^2=686$

Similarly

Use distance formula

$BC^2=(25-10)^2+(-41-20)^2+(5-30)^2$
$=225+3721+625$
$=4571$

And, similarly

$CA^2 = (3-25)^2 + (6+41)^2 + (9-5)^2$

$= 484 + 2209 + 16$

$= 2709$

In Right angle tringle,

$(\text{ Hypotenuse })^2 = (\text{ Height })^2 + (\text{ Base })^2$

To find that $CA^2 + AB^2 \neq BC^2$.

Therefore, the triangle ABC is not a right angled triangle.

Example 6

Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points $(3,4,5)$ and $(-1,3,-7)$, respectively.

Solution

Let the coordinates of point P be (x, y, z).

Given,

$A(3,4,5)$

$B(-1,3,-7)$

Using the distance formula

Here,

$PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$
$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$, we have

Putting the values

$(x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = 2k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 50 + 59 = 2k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 50 - 59$

Rearrange the equation

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$

EXERCISE 12.2

1. Find the distance between the following pairs of points:

(i) $(2,3,5)$ **and** $(4,3,1)$

(ii) $(-3,7,2)$ **and** $(2,4,-1)$

(iii) $(-1,3,-4)$ **and** $(1,-3,4)$

(iv) $(2,-1,3)$ **and** $(-2,1,3)$.

Solution

(i) The distance between the points $\mathrm{P}(2,3,5)$ and $Q(4,3,1)$ by distance formula is,

$$\mathrm{PQ}=\sqrt{(4-2)^2+(3-3)^2+(1-5)^2}$$
$$=\sqrt{4+0+16}$$
$$=\sqrt{20}$$
$$=\sqrt{4\times 5}$$
$$=2\sqrt{5}$$

(ii) The distance between the points $\mathrm{P}(-3,7,2)$ and $Q(2,4,-1)$ is

$$\mathrm{PQ}=\sqrt{(2+3)^2+(4-7)^2+(-1-2)^2}$$
$$=\sqrt{25+9+9}$$
$$=\sqrt{43}$$

(iii) The distance between the points $\mathrm{P}(-1,3,-4)$ and $Q(1,-3,4)$ is

$$\mathrm{PQ}=\sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$$
$$=\sqrt{4+36+64}$$
$$=\sqrt{104}$$
$$=\sqrt{4\times 26}$$
$$=2\sqrt{26}$$

(iv) The distance between the points $\mathrm{P}(2,-1,3)$ and $Q(-2,1,3)$ is

$$\mathrm{PQ}=\sqrt{(-2-2)^2+(1+1)^2+(3-3)^2}$$
$$=\sqrt{16+4+0}$$
$$=\sqrt{20}$$
$$=\sqrt{4\times 5}$$
$$=2\sqrt{5}$$

2. Show that the points $(-2,3,5),(1,2,3)$ and $(7,0,-1)$ are collinear.

Solution

Let $P(-2,3,5), Q(1,2,3), R(7,0,-1)$ be the given points.

Distance between points

$$PQ=\sqrt{(1+2)^2+(2-3)^2+(3-5)^2}$$
$$=\sqrt{9+1+4}$$
$$=\sqrt{14}$$

And,

$$QR=\sqrt{(7-1)^2+(0-2)^2+(-1-3)^2}$$
$$=\sqrt{36+4+16}$$
$$=\sqrt{56}$$
$$=\sqrt{4\times 14}$$
$$=2\sqrt{14}$$

Now,

$$PR=\sqrt{(7+2)^2+(0-3)^2+(-1-5)^2}$$
$$=\sqrt{81+9+36}$$
$$=\sqrt{126}$$
$$=\sqrt{9\times 14}$$
$$=3\sqrt{14}$$

So,

$$PQ+QR=\sqrt{14}+2\sqrt{14}$$
$$=(1+2)\sqrt{14}$$
$$=3\sqrt{14}=PR$$

Therefore, points P, Q, R are collinear.

3. Verify the following:

(i) $(0,7,-10),(1,6,-6)$ **and** $(4,9,-6)$ **are the vertices of an isosceles triangle.**

(ii) $(0,7,10),(-1,6,6)$ **and** $(-4,9,6)$ **are the vertices of a right angled triangle.**

(iii) $(-1,2,1),(1,-2,5),(4,-7,8)$ **and** $(2,-3,4)$ **are the vertices of a parallelogram.**

Solution

(i) Let $A(0,7,-10), B(1,6,-6)$ and $C(4,9,-6)$ be the given points.

$AB=\sqrt{(1-0)^2+(6-7)^2+(-6+10)^2}$

$=\sqrt{1+1+16}$

$=\sqrt{18}$

$=\sqrt{9\times 2}$

$=3\sqrt{2}$

$BC=\sqrt{(4-1)^2+(9-6)^2+(-6+6)^2}$

$=\sqrt{9+9+0}$

$=\sqrt{18}$

$=\sqrt{9\times 2}$

$=3\sqrt{2}$

$AC=\sqrt{(4-0)^2+(9-7)^2+(-6+10)^2}$

$=\sqrt{16+4+16}$

$=\sqrt{36}$

$=6$

Since $AB=BC=3\sqrt{2}$,

If the two sides of a triangle is same

So, triangle ABC is isosceles.

(ii) Let $A(0,7,10), B(-1,6,6)$ and $C(-4,9,6)$ be the given points.

Distance formula

$AB=\sqrt{(-1-0)^2+(6-7)^2+(6-10)^2}$

$=\sqrt{1+1+16}$

$=\sqrt{18}$

$=3\sqrt{2}$

$BC=\sqrt{(-4+1)^2+(9-6)^2+(6-6)^2}$

$=\sqrt{9+9+0}$

$=\sqrt{18}$

$=3\sqrt{2}$

$AC=\sqrt{(-4-0)^2+(9-7)^2+(6-10)^2}$

$=\sqrt{16+4+16}$

$=\sqrt{36}$

$=6$

Since $AB^2+BC^2=18+18=36=AC^2$

$AB^2+BC^2=AC^2$

$\triangle ABC$ is right angled at B.

Therefore, ABC is a right-angled isosceles triangle.

(iii) Let $A(-1,2,1), B(1,-2,5), C(4,-7,8)$ and $D(2,-3,4)$ be the given points.

The distance formula

$AB=\sqrt{(1+1)^2+(-2-2)^2+(5-1)^2}$

$=\sqrt{4+16+16}$

$=\sqrt{36}$

$=6$

$DC=\sqrt{(4-2)^2+(-7+3)^2+(8-4)^2}$

$=\sqrt{4+16+16}$

$=\sqrt{36}$

$=6$

$AD=\sqrt{(2+1)^2+(-3-2)^2+(4-1)^2}$

$=\sqrt{9+25+9}$

$=\sqrt{43}$

$BC=\sqrt{(4-1)^2+(-7+2)^2+(8-5)^2}$

$=\sqrt{9+25+9}$

$=\sqrt{43}$

Since $AB=DC=6$ and $AD=BC=\sqrt{43}$

$\Rightarrow$ The opposite sides of quadrilateral $ABCD$ are equal.

$\Rightarrow$ Quadrilateral $ABCD$ is a parallelogram.

4. Find the equation of the set of points which are equidistant from the points $(1,2,3)$ and $(3,2,-1)$.

Solution

Let $\mathrm{P}(x, y, z)$ be equidistant from the points $\mathrm{A}(1,2,3)$ and $\mathrm{B}(3,2,-1)$

So, $\mathrm{AP}=\mathrm{BP}$

$\mathrm{AP}^2=\mathrm{BP}^2$

$(x-1)^2+(y-2)^2+(z-3)^2=(x-3)^2+(y-2)^2+(z+1)^2$

$\left(x^2-2x+1\right)+\left(y^2-4y+4\right)+\left(z^2-6z+9\right)=\left(x^2-6x+9\right)+\left(y^2-4y+4\right)+\left(z^2+2z+1\right)$

Rearrange the equation

$-2x+6x-6z-2z+14-14=0$

$4x-8z=0$

Dividing every term by 4,

$x-2z=0$ which is the required equation of the set of points $\mathrm{P}(x, y, z)$

5. Find the equation of the set of points P, the sum of whose distances from $\mathrm{A}(4,0,0)$ and $\mathrm{B}(-4,0,0)$ is equal to 10.

Solution

Given points are $\mathrm{A}(4,0,0)$ and $\mathrm{B}(-4,0,0)$.

Let the coordinates of P be (x, y, z).

Given: $\mathrm{AP}+\mathrm{BP}=10$

Applying distance formula

$\Rightarrow \sqrt{(x-4)^2+(y-0)^2+(z-0)^2}+\sqrt{(x+4)^2+(y-0)^2+(z-0)^2}=10$

$\Rightarrow \sqrt{(x-4)^2+y^2+z^2}+\sqrt{(x+4)^2+y^2+z^2}=10$

Shifting one square root term to R.H.S;

$\Rightarrow \sqrt{(x-4)^2+y^2+z^2}=10-\sqrt{(x+4)^2+y^2+z^2}$

Squaring both sides,

$(x-4)^2+y^2+z^2=100+(x+4)^2+y^2+z^2-20\sqrt{(x+4)^2+y^2+z^2}$

$x^2+16-8x=100+x^2+16+8x-20\sqrt{(x+4)^2+y^2+z^2}$

$\Rightarrow -16x-100=-20\sqrt{(x+4)^2+y^2+z^2}$

Dividing both sides by (-4),

$$4x+25=5\sqrt{(x+4)^2+y^2+z^2}$$

Squaring both sides again,

$$16x^2+200x+625=25\left[x^2+8x+16+y^2+z^2\right]$$

$$16x^2+200x+625=25x^2+200x+400+25y^2+25z^2$$

$$\Rightarrow -9x^2-25y^2-25z^2=-225$$

multiplying by -1,

$$9x^2+25y^2+25z^2=225$$

Which is required equation (i.e., locus) of set of points P.

Example 7

Find the coordinates of the point which divides the line segment joining the points $(1,-2,3)$ and $(3,4,-5)$ in the ratio $2:3$ (i) internally, and (ii) externally.

Solution

(i) Let $P(x,y,z)$ be the point which divides line segment joining $A(1,-2,3)$ and $B(3,4,-5)$ internally in the ratio $2:3$.

So, Applying the coordinates of mid-point

For x axis

$$x=\frac{2(3)+3(1)}{2+3}=\frac{9}{5},$$

For y axis

$$y=\frac{2(4)+3(-2)}{2+3}=\frac{2}{5},$$

For z Axis

$$z=\frac{2(-5)+3(3)}{2+3}=\frac{-1}{5}$$

Thus, the required point of the coordinate is

$$\left(\frac{9}{5},\frac{2}{5},\frac{-1}{5}\right)$$

(ii) Let $P(x, y, z)$ be the point which divides segment joining $A(1,-2,3)$ and B $(3,4,-5)$ externally in the ratio $2:3$.

Then, Applying the coordinate mid point rule

For x axis

$$x=\frac{2(3)+(-3)(1)}{2+(-3)}=-3,$$

For x axis

$$y=\frac{2(4)+(-3)(-2)}{2+(-3)}=-14,$$

For z axis

$$z=\frac{2(-5)+(-3)(3)}{2+(-3)}=19$$

So, the required point of the coordinate is $(-3,-14,19)$.

Example 8

Using section formula, prove that the three points $(-4,6,10),(2,4,6)$ **and** $(14,0,-2)$ **are collinear.**

Solution

Let $A(-4,6,10), B(2,4,6)$ and $C(14,0,-2)$ be the given points.

Let the point P divides AB in the ratio $k:1$.

Then coordinates of the point P are

$$\left(\frac{2k-4}{k+1},\frac{4k+6}{k+1},\frac{6k+10}{k+1}\right)$$

Check whether the value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1}=14$,

The value of $k=-\frac{3}{2}$

When $k=-\frac{3}{2}$,

then $\frac{4k+6}{k+1}=\frac{4\left(-\frac{3}{2}\right)+6}{-\frac{3}{2}+1}=0$

and

$$\frac{6k+10}{k+1}=\frac{6\left(-\frac{3}{2}\right)+10}{-\frac{3}{2}+1}=-2$$

So, C$(14,0,-2)$ is a point is divided AB by externally in the ratio $3:2$ and is same as P .

Hence A,B,C are collinear.

Example 9

Find the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1, z_1),(x_2, y_2, z_2)$ and (x_3, y_3, z_3).

Solution

Let ABC be the triangle.

Let the coordinates of the vertices A, B, C be $(x_1, y_1, z_1),(x_2, y_2, z_2)$ and (x_3, y_3, z_3), respectively.

Let D be the mid-point of BC .

Hence coordinates of D are

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2}\right)$$

Let G be the centroid of the triangle.

It divides the median AD in the ratio $2:1$.

Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1},\frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1},\frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

Cancel the common term

or

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

Example 10

Find the ratio in which the line segment joining the points $(4,8,10)$ and $(6,10,-8)$ is divided by the $\text{YZ}-$plane.

Solution

Let $YZ-$plane divided the line segment joins A $(4,8,10)$ and B $(6,10,-8)$ at $\text{P}(x,y,z)$ in the ratio $k:1$.

Then the coordinates of P are

$$\left(\frac{4+6k}{k+1},\frac{8+10k}{k+1},\frac{10-8k}{k+1}\right)$$

Since P lies on the YZ-plane, its x-coordinate is zero,

That is $\frac{4+6k}{k+1}=0$

Rearrange the equation

or

$$k=-\frac{2}{3}$$

So, YZ-plane divided AB by externally in the ratio $2:3$.

EXERCISE 12.3

1. Find the coordinates of the point which divides the line segment joining the points $(-2,3,5)$ and $(1,-4,6)$ in the ratio

(i) $2:3$ internally,

(ii) $2:3$ externally.

Solution

(i) Let $\text{P}(x,y,z)$ be the point which divided the line segment joining $\text{A}(-2,3,5)$ and $\text{B}(1,-4,6)$ internally in the ratio $2:3$.

Then, by section formula;

For coordinate of x

$$x=\frac{2(1)+3(-2)}{2+3}=-\frac{4}{5},$$

For coordinate of y

$$y=\frac{2(-4)+3(3)}{2+3}=\frac{1}{5}$$

For coordinate of z

$$z=\frac{2(6)+3(5)}{2+3}=\frac{27}{5}$$

Hence, the required point is $\left(-\frac{4}{5},\frac{1}{5},\frac{27}{5}\right)$.

(ii) Let $P(x,y,z)$ be the point which divides the line segment joining $A(-2,3,5)$ and $B(1,-4,6)$ externally in the ratio $2:3$.

Then, by section formula

For coordinate of x

$$x=\frac{2(1)-3(-2)}{2-3}=-8,$$

For coordinate y

$$y=\frac{2(-4)-3(3)}{2-3}=17$$

For coordinate z

$$z=\frac{2(6)-3(5)}{2-3}=3$$

Hence, the required point is $(-8,17,3)$.

2. Given that $P(3,2,-4), Q(5,4,-6)$ and $R(9,8,-10)$ are collinear. Find the ratio in which Q divides PR.

Solution

Let Q divide PR in the ratio $k:1$,

then the coordinates of Q are $\left(\frac{9k+3}{k+1},\frac{8k+2}{k+1},\frac{-10k-4}{k+1}\right)$

But the coordinates of Q are given to be $(5,4,-6)$ Equating coordinates,

$$\frac{9k+3}{k+1}=5 \text{ and } \frac{8k+2}{k+1}=4 \text{ and } \frac{-10k-4}{k+1}=-6$$

Solving any one of these equations for k, say first,

we have on cross-multiplication,

$9k+3=5k+5$

$4k=2$.

$\therefore k=\frac{1}{2}$

Hence, Q divides PR in the ratio $\frac{1}{2}:1=1:2$.

3. Find the ratio in which the YZ-plane divides the line segment formed by joining the points $(-2,4,7)$ and $(3,-5,8)$.

Solution

Let the YZ-plane divide the line segment joining $A(-2,4,7)$ and $B(3,-5,8)$ at $P(x,y,z)$ in the ratio $k:1$. Then the coordinates of P are

$$\left(\frac{3k-2}{k+1},\frac{-5k+4}{k+1},\frac{8k+7}{k+1}\right)$$

Since P lies on the $YZ-$plane, its x-coordinate is zero,

That is $\frac{3k-2}{k+1}=0$

cross-multiplying,

$3k-2=0$

$\Rightarrow 3k=2$

or $k=\frac{2}{3}(+\text{ve})$

Therefore, YZ-plane divides AB internally in the ratio $\frac{2}{3}:1$

That is $2:3$.

4. Using section formula, show that the points A $(2,-3,4), B(-1,2,1)$ and $C\left(0,\frac{1}{3},2\right)$ are collinear.

Solution

Let the point C divide the line joining the points A and B in the ratio $k:1$.

Then the coordinates of C are

$$\left(\frac{-k+2}{k+1}, \frac{2k-3}{k+1}, \frac{k+4}{k+1}\right)$$

But coordinates of C are given $\left(0, \frac{1}{3}, 2\right)$

Equating coordinates of C given and obtained,

$$\frac{-k+2}{k+1} = 0$$

cross-multiplying

$$-k+2=0$$

$$k=2$$

Now,

$$\frac{2k-3}{k+1} = \frac{1}{3}$$

cross-multiplying

$$6k-9=k+1$$

$$5k=10$$

$$k=2$$

And,

$$\frac{k+4}{k+1} = 2$$

$$k+4=2k+2$$

$$2=k$$

The value of k from all the three equations is equals

$\therefore$ Points A, B and C are collinear (and the point C divides AB in the ratio $2:1$).

5. Find the coordinates of the points which trisect the line segment joining the points $P(4,2,-6)$ and $Q(10,-16,6)$

Solution

Given: points $P(4,2,-6)$ and $Q(10,-16,6)$.

Let take two points R and S within the segment PQ

such that $PR = RS = SQ$.

Then points R and S are called points of trisection of the segment PQ

Therefore, One point of trisection R divides the joins of P and Q internally in the ratio $1:2(=1+1)$.

Point P is $(4,2,-6)$ and Q is $(10,-16,6)$

Coordinates of point R are

$$R\left[\frac{1(10)+2(4)}{1+2},\frac{1(-16)+2(2)}{1+2},\frac{1(6)+2(-6)}{1+2}\right]$$

So, the $R\left(\frac{18}{3},\frac{-12}{3},\frac{-6}{3}\right)=(6,-4,-2)$

Again, the second point of trisection S divides the joins of P and Q internally in the ratio of $2(=1+1):1$.

Coordinates of point S are

$$S\left[\frac{2(10)+1(4)}{2+1},\frac{2(-16)+1(2)}{2+1},\frac{2(6)+1(-6)}{2+1}\right]$$

$$S\left(\frac{24}{3},\frac{-30}{3},\frac{6}{3}\right)=(8,-10,2)$$

Point S is the mid-point of RQ

The coordinate of point is $(8,-10,2)$

Example 11

Show that the points A $(1,2,3), B(-1,-2,-1), C(2,3,2)$ and $D(4,7,6)$ are the vertices of a parallelogram $ABCD$, but it is not a rectangle.

Solution

To show $ABCD$ is a parallelogram need to show opposite side are equal

By using the distance formula

$$AB=\sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2}$$
$$=\sqrt{4+16+16}$$
$$=6$$

So,

$$BC = \sqrt{(2+1)^2 + (3+2)^2 + (2+1)^2}$$
$$= \sqrt{9+25+9}$$
$$= \sqrt{43}$$

Now,

$$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2}$$
$$= \sqrt{4+16+16}$$
$$= 6$$

And

$$DA = \sqrt{(1-4)^2 + (2-7)^2 + (3-6)^2}$$
$$= \sqrt{9+25+9}$$
$$= \sqrt{43}$$

Since $AB = CD$ and $BC = AD$,

$ABCD$ is a parallelogram.

Now, it is required to prove that $ABCD$ is not a rectangle. For this, we show that diagonals AC and BD are unequal.

Same as using the distance

$$AC = \sqrt{(2-1)^2 + (3-2)^2 + (2-3)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$

And

$$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2}$$
$$= \sqrt{25+81+49}$$
$$= \sqrt{155}$$

Since $AC \neq BD$,

$ABCD$ is not a rectangle.

Example 12

Find the equation of the set of the points P such that its distances from the points A $(3,4,-5)$ and B $(-2,1,4)$ are equal.

Solution

If $P(x, y, z)$ be any point such that $PA = PB$.

Given that,

A $(3, 4, -5)$ and B $(-2, 1, 4)$ are equal.

Now, by applying the distance formula

$$\sqrt{(x-3)^2+(y-4)^2+(z+5)^2}=\sqrt{(x+2)^2+(y-1)^2+(z-4)^2}$$

or $(x-3)^2+(y-4)^2+(z+5)^2=(x+2)^2+(y-1)^2+(z-4)^2$

or $x^2+9-6x+y^2+4-8y+z^2+25+10z=x^2+4+8x+y^2+1-2y+z^2+16-8z$

or $10x+6y-18z-29=0$.

Example 13

The centroid of a triangle ABC is at the point $(1,1,1)$. If the coordinates of A and B are $(3,-5,7)$ and $(-1,7,-6)$, respectively, find the coordinates of the point C.

Solution

Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be $(1,1,1)$.

Coordinates of A is $(3,-5,7)$

Coordinates of B is $(-1,7,-6)$

Then

$$\frac{x+3-1}{3}=1$$

$.x=1;,$

And,

$$\frac{y-5+7}{3}=1$$

That is $y=1$

$$\frac{z+7-6}{3}=1,$$

That is $z=2$

Therefore, coordinates of C are $(1,1,2)$.

Miscellaneous Exercise on Chapter 12

1. Three vertices of a parallelogram $ABCD$ are $A(3,-1,2), B(1,2,-4)$ and $C(-1,1,2)$. Find the coordinates of the fourth vertex.

Solution

Let the coordinates of D be (x,y,z).

Given,

$A(3,-1,2), B(1,2,-4)$ and $C(-1,1,2)$.

M the point of intersection of the diagonals of the parallelogram

The diagonals of a $\|^{gm}$ bisect each other.

Since M is the mid-point AC,

Therefore, Coordinates of M are $\left(\frac{3-1}{2},\frac{-1+1}{2},\frac{2+2}{2}\right)=(1,0,2)$

Also M is the mid-point of BD

Coordinates of M are $\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$

Equating corresponding entries,

$1=\frac{x+1}{2}$,

$0=\frac{y+2}{2}$

and $2=\frac{z-4}{2}$

Now, cross-multiplying

$x+1=2, y+2=0$ and $z-4=4$

$x=1,$

$y=-2$

and $z=8$

Hence, vertex D is $(1,-2,8)$.

2. Find the lengths of the medians of the triangle with vertices A $(0,0,6), B(0,4,0)$ and $(6,0,0)$.

Solution

Let D, E, F be the midpoints of sides BC, CA, AB respectively.

Then

$$D=\left(\frac{0+6}{2},\frac{4+0}{2},\frac{0+0}{2}\right)=(3,2,0)$$

So, Length of median AD

$$=\sqrt{(3-0)^2+(2-0)^2+(0-6)^2}$$
$$=\sqrt{9+4+36}$$
$$=\sqrt{49}$$
$$=7$$

So, coordinates of E

$$E=\left(\frac{6+0}{2},\frac{0+0}{2},\frac{0+6}{2}\right)=(3,0,3)$$

Therefore, Length of median BE

$$=\sqrt{(3-0)^2+(0-4)^2+(3-0)^2}$$

$$=\sqrt{9+16+9}=\sqrt{34}$$

Coordinate of point F

$$F=\left(\frac{0+0}{2},\frac{0+4}{2},\frac{6+0}{2}\right)=(0,2,3)$$

Then, Length of median CF

$$=\sqrt{(0-6)^2+(2-0)^2+(3-0)^2}$$

$$=\sqrt{36+4+9}=\sqrt{49}=7$$

The length of medians of triangle is $AD=7$, $BE=\sqrt{34}$, $CF=7$

3. If the origin is the centroid of the triangle PQR with vertices $P(2a,2,6)$, $Q(-4,3b,-10)$ and $R(8,14,2c)$, then find the values of a, b and c.

Solution

By formula; centroid of the triangle PQR with vertices $P(2a,2,6), Q(-4,3b,-10)$ and $R(8,14,2c)$ is

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

Substitute the values

$$\left(\frac{2a-4+8}{3},\frac{2+3b+14}{3},\frac{6-10+2c}{3}\right)$$

$$\left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)= \text{origin } =(0,0,0)$$

Equating coordinates,

$$\frac{2a+4}{3}=0$$

$$2a+4=0$$

Rearrange the equation

$$2a=-4$$

$$a=-2$$

Equating coordinates,

$$\frac{3b+16}{3}=0$$

$$3b+16=0$$

Rearrange the equation

$$3b=-16$$

$$b=-\frac{16}{3}$$

Equating coordinates

$$\frac{2c-4}{3}=0$$

$$2c-4=0$$

$$2c=4$$

$$c=2.$$

The value of $a=-2, b=\frac{-16}{3}, c=2$

4. Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point $P(3,-2,5)$

Solution

Let the required point on y-axis be $A(0,y,0)$.

Given: Point $P(3,-2,5)$ and distance $AP = 5\sqrt{2}$

$\Rightarrow \sqrt{(0-3)^2+(y+2)^2+(0-5)^2} = 5\sqrt{2}$

Squaring, $9+(y+2)^2+25=50$

$(y+2)^2 = 50-34$

$\Rightarrow (y+2)^2 = 16$

$y+2 = \pm 4$

$\Rightarrow y = -2 \pm 4$

$\Rightarrow y = 2 \text{ or } -6$

Therefore, the required points are $(0,y,0)=(0,2,0)$ and $(0, -6, 0)$

5. A point R with x-coordinate 4 lies on the line segment joining the points $P(2,-3,4)$ and $Q(8,0,10)$. Find the coordinates of the point R.

Solution

Since R lies on PQ, therefore, R divides PQ in some ratio, say $k:1$.

Therefore, by section formula, R has coordinates $\left(\dfrac{8k+2}{k+1}, \dfrac{0-3}{k+1}, \dfrac{10k+4}{k+1}\right)$

x-coordinate of R is given to be 4

Therefore, $\dfrac{8k+2}{k+1} = 4$

$\Rightarrow 8k+2 = 4(k+1)$

$8k+2 = 4k+4$

Rearrange the equation

$\Rightarrow 4k = 2$

$k = \dfrac{1}{2}$

Putting this value of k in equation, the required coordinates of R are

$$\left(4,\frac{-3}{1/2+1},\frac{10\times1/2+4}{1/2+1}\right)=\left(4,\frac{-3}{\left(\frac{3}{2}\right)},\frac{9}{\left(\frac{3}{2}\right)}\right)$$

$$=(4,-2,6)$$

The coordinates of the point $R=(4,-2,6)$

6.If A and B be the points $(3,4,5)$ and $(-1,3,-7)$, respectively, find the equation of the set of points P such that $PA^2+PB^2=k^2$, where k is a constant.

Solution

Let the coordinates of P be (x,y,z).

Given: Points $A(3,4,5)$ and $B(-1,3,-7)$ and $PA^2+PB^2=k^2$

$$\left[(x-3)^2+(y-4)^2+(z-5)^2\right]+\left[(x+1)^2+(y-3)^2+(z+7)^2\right]=k^2$$

Rearrange the equation

$$\left(x^2-6x+9+y^2-8y+16+z^2-10z+25\right)+\left(x^2+2x+1+y^2-6y+9+z^2+14z+49\right)=k^2$$

Rearrange the equation

$$2x^2+2y^2+2z^2-4x-14y+4z+109=k^2$$

$$2\left(x^2+y^2+z^2-2x-7y+2z\right)=k^2-109$$

Dividing by 2,

$$x^2+y^2+z^2-2x-7y+2z=\frac{k^2-109}{2}$$

Which is the required equation of set of points P .