Chapter 13: Limit and Derivative

Example 1

Find the limits: (i) $\lim_{x \to 1}\left[x^3 - x^2 + 1\right]$

(ii) $\lim_{x \to 3}[x(x+1)]$

(iii) $\lim_{x \to -1}\left[1 + x + x^2 + \ldots + x^{10}\right]$.

Solution

Given that

(i) $\lim_{x \to 1}\left[x^3 - x^2 + 1\right]$

Put $x = 1$

$= 1^3 - 1^2 + 1$

$= 1$

(ii) $\lim_{x \to 3}[x(x+1)]$

Put the value for x

$= 3(3+1)$

$= 3(4) = 12$

(iii) $\lim_{x \to -1}\left[1 + x + x^2 + \ldots + x^{10}\right]$

Put $x = 1$

$= 1 + (-1) + (-1)^2 + \ldots + (-1)^{10}$

$= 1 - 1 + 1 \ldots + 1 = 1$

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Example 2

(i) $\lim_{x \to 1}\left[\frac{x^2 + 1}{x + 100}\right]$

(ii) $\lim_{x \to 2}\left[\frac{x^3 - 4x^2 + 4x}{x^2 - 4}\right]$

(iii) $\lim_{x\to 2}\left[\frac{x^2-4}{x^3-4x^2+4x}\right]$

(iv) $\lim_{x\to 2}\left[\frac{x^3-2x^2}{x^2-5x+6}\right]$

(v) $\lim_{x\to 1}\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$.

Solution

Given that

(i) We have $\lim_{x\to 1}\frac{x^2+1}{x+100}=\frac{1^2+1}{1+100}=\frac{2}{101}$

(ii) Put $x=2$

Hence $\lim_{x\to 2}\frac{x^3-4x^2+4x}{x^2-4}=\lim_{x\to 2}\frac{x(x-2)^2}{(x+2)(x-2)}$

$=\lim_{x\to 2}\frac{x(x-2)}{(x+2)}$

$=\frac{2(2-2)}{2+2}=\frac{0}{4}=0$

as $x\neq 2$

(iii) Put $x=2$

$\lim_{x\to 2}\frac{x^2-4}{x^3-4x^2+4x}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x(x-2)^2}$

$=\lim_{x\to 2}\frac{(x+2)}{x(x-2)}=\frac{2+2}{2(2-2)}=\frac{4}{0}$

which is not defined.

(iv)Put $x=2$

Hence $\lim_{x\to 2}\frac{x^3-2x^2}{x^2-5x+6}=\lim_{x\to 2}\frac{x^2(x-2)}{(x-2)(x-3)}$

$=\lim_{x\to 2}\frac{x^2}{(x-3)}=\frac{(2)^2}{2-3}=\frac{4}{-1}=-4$

(v) Write in rational form

$$\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\left[\frac{x-2}{x(x-1)}-\frac{1}{x\left(x^2-3x+2\right)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\frac{x^2-4x+3}{x(x-1)(x-2)}$$

Hence $\lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

$$=\lim_{x\to 1}\frac{x^2-4x+3}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}=\frac{1-3}{1(1-2)}=2$$

Example 3

Evaluate:

(i) $\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}$

(ii) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

Solution

(i) We have

$$\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}=\lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\div\frac{x^{10}-1}{x-1}\right]$$

$$=\lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\right]\div\lim_{x\to 1}\left[\frac{x^{10}-1}{x-1}\right]$$

$=15(1)^{14}\div 10(1)^{9}$ (by the theorem above)

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$=15\div 10=\frac{3}{2}$

(ii) Put $y=1+x$,

so $y\to 1$ as $x\to 0$.

Then

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$$

$$=\lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$$

$$=\frac{1}{2}(1)^{\frac{1}{2}-1}=\frac{1}{2}$$

Example 4

Evaluate:

(i) $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}$

(ii) $\lim_{x\to 0}\frac{\tan x}{x}$

Solution

Given that

(i) $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\left[\frac{\sin 4x}{4x}\cdot\frac{2x}{\sin 2x}\cdot 2\right]$

$$=2.\lim_{x\to 0}\left[\frac{\sin 4x}{4x}\right]\div\left[\frac{\sin 2x}{2x}\right]$$

$$=2.\lim_{4x\to 0}\left[\frac{\sin 4x}{4x}\right]\div\lim_{2x\to 0}\left[\frac{\sin 2x}{2x}\right]$$

Simplify

$=2.1.1=2(\text{as } x\to 0, 4x\to 0 \text{ and } 2x\to 0)$

(ii) We have $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1.1=1$

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$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}$.

Exercise 13.1

Question 1:

Evaluate the Given $\lim_{x \to 3} x + 3$ limit:

Answer 1:

Given that

$$\lim_{x \to 3} x + 3$$

Put $x = 3$

$$\lim_{x \to 3} x + 3 = 3 + 3 = 6$$

Question 2:

Evaluate the Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7} \right)$

Answer 2:

Given that

$$\lim_{x \to \pi} \left(x - \frac{22}{7} \right)$$

Put $x = \pi$

$$\lim_{x \to \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$$

Question 3:

Evaluate the Given limit: $\lim_{r \to 1} r^2$

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Answer 3:

Given that

$$\lim_{r\to 1} r^2$$

Put $r=1$

$$\lim_{r\to 1} \pi r^2 = \pi(1)^2 = \pi$$

Question 4:

Evaluate the Given limit: $\lim_{x\to 4} \frac{4x+3}{x-2}$

Answer 4:

Given that

$$\lim_{x\to 4} \frac{4x+3}{x-2}$$

Put $x=4$

$$\lim_{x\to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2}$$

$$= \frac{16+3}{2} = \frac{19}{2}$$

Question 5:

Evaluate the Given limit: $\lim_{x\to -1} \frac{x^{10}+x^5+1}{x-1}$

Answer 5:

Given that

$$\lim_{x\to -1} \frac{x^{10}+x^5+1}{x-1}$$

Put $x=-1$

$$\lim_{x\to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$$

Question 6:

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Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Answer 6:

Given

$$\lim_{x\to 0}\frac{(x+1)^5-1}{x}$$

Put $x+1=y$ so that $y\to 1$ as $x\to 0$

Accordingly, $\lim_{x\to 0}\frac{(x+1)^5-1}{x}=\lim_{y\to 1}\frac{y^5-1}{y-1}$

$$=\lim_{y\to 1}\frac{y^5-1^5}{y-1}$$

$$=5.1^{5-1}\quad\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=5$$

$$\therefore \lim_{x\to 0}\frac{(x+5)^5-1}{x}=5$$

Question 7:

Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Answer 7:

Given that

$$\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$$

At $x=2$,

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{3x+5}{x+2}$$

$$=\frac{3(2)+5}{2+2}$$

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$=\frac{11}{4}$

Question 8:

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer 8:

Given that

$$\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$$

At $x=2$

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3} \quad =\lim_{x\to 3}\frac{(x-3)(x+3)\left(x^2+9\right)}{(x-3)(2x+1)}$$

$$=\lim_{x\to 3}\frac{(x+3)\left(x^2+9\right)}{2x+1}$$

$$=\frac{(3+3)\left(3^2+9\right)}{2(3)+1}$$

simplify

$$=\frac{6\times 18}{7}$$

$$=\frac{108}{7}$$

Question 9:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Answer 9:

Given

$$\lim_{x\to 0}\frac{ax+b}{cx+1}$$

Put $x=0$

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Question 10:

Evaluate the Given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Answer 10:

Given that

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At $z=1$

Put $z^{\frac{1}{6}}=x$

so that $z\to 1$ as $x\to 1$

Accordingly, $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{\frac{1}{6}-1}=\lim_{x\to 1}\frac{x^2-1}{x-1}$

$$=\lim_{x\to 1}\frac{x^2-1^2}{x-1}$$

$$=2.1^{2-1}\quad\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=2$$

$$\therefore \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=2$$

Question 11:

Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Answer 11:

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Given that

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$

simplify

$$=\frac{a+b+c}{a+b+c}$$

$$=1$$

Then

$$[a+b+c\neq 0]$$

Question 12:

Evaluate the Given limit: $\lim_{x\to -2}\frac{x^{+}2}{x+2}$

Answer 12:

Given

$$\lim_{x\to -2}\frac{x^{+}2}{x+2}$$

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At $x=-2$

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$

$$=\lim_{x\to -2}\frac{1}{2x}$$

Apply values

$$=\frac{1}{2(-2)}=\frac{-1}{4}$$

Question 13:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

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Answer 13:

Given that

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

At $x=0$

Now, $\lim_{x\to 0}\frac{\sin ax}{bx}$

$$=\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$

$$=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\left(\frac{a}{b}\right)$$

$$=\frac{a}{b}\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)$$

Apply value

$$=\frac{a}{b}\times 1$$

$$=\frac{a}{b}$$

Question 14:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a,b\neq 0$

Answer 14:

Given that

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a,b\neq 0$$

At $x=0$

Now, $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

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simplify

$$=\left(\frac{a}{b}\right)\times\frac{\lim\limits_{\alpha x\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{bx\to 0}\left(\frac{\sin bx}{bx}\right)}$$

$$=\left(\frac{a}{b}\right)\times\frac{1}{1}$$

$$=\frac{a}{b}$$

Question 15:

Evaluate the Given limit: $\lim\limits_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Answer 15:

Given that

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that $x\to\pi\Rightarrow(\pi-x)\to 0$

$$\therefore\lim_{x\to z}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(z-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

simplify

$$=\frac{1}{\pi}\times 1$$

$$=\frac{1}{\pi}$$

Question 16:

Evaluate the given limit: $\lim\limits_{x\to 0}\frac{\cos x}{\pi-x}$

Answer 16:

Given that

$$\lim_{x\to 0}\frac{\cos x}{\pi-x}$$

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Put $x=0$

$$\lim_{x\to 0}\frac{\cos x}{\pi - x}=\frac{\cos 0}{\pi - 0}=\frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Answer 17:

Given

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$$

At $x=0$,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

simplify

$$=4\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)$$

$$\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}$$

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$$=4\frac{1^2}{1^2}$$

$$=4$$

Question 18:

Evaluate the Given limit: $\lim\limits_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer 18:

Given

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At $x=0$,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b^{x\to 0}}\lim\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

simplify

$$=\frac{1}{b}\times\frac{1}{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)^{\times}\lim\limits_{x\to 0}(a+\cos x)}$$

$$=\frac{1}{b}\times(a+\cos 0)$$

$$=\frac{a+1}{b}$$

Question 19:

Evaluate the Given limit: $\lim\limits_{x\to 0}x\sec x$

Answer 19:

Given

$$\lim_{x\to 0}x\sec x$$

Put $x=0$

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$$\lim_{x\to 0} x\sec x = \lim_{x\to 0}\frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

Question 20:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx} a, b, a + b \neq 0$

Answer 20:

Given

$$\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx} a, b, a + b \neq 0$$

At $x = 0$,

$$\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{\left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)\times \lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx\left(\lim_{bx\to 0}\frac{\sin bx}{bx}\right)}$$

simplify

$$= \frac{\lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx}$$

$$= \frac{\lim_{x\to 0}(ax + bx)}{\lim_{x\to 0}(ax + bx)}$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

Question 21:

Evaluate the Given limit: $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

Answer 21:

Given

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$

At $x = 0$,

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim\limits_{x\to 0}\frac{1-\cos x}{x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{0}{1}$$

$$= 0$$

Question 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer 22:

Given that

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Put $x = \frac{\pi}{2}$,

$x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$=\lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$

$$=\lim_{y\to 0}\frac{\tan 2y}{y}$$

$$=\lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

$$=\lim_{y\to 0}\left(\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}\right)$$

$$=\left(\lim_{2y\to 0}\frac{\sin 2y}{2y}\right)\times\lim_{y\to 0}\left(\frac{2}{\cos 2y}\right)$$

simplify

$$=1\times\frac{2}{\cos 0}$$

$$=1\times\frac{2}{1}$$

$$=2$$

Question 23:

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$

Answer 23:

The given function is $f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}[2x+3]=2(0)+3=3$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}3(x+1)=3(0+1)=3$$

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$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$

Question 24:

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Answer 24:

The given function is

$$f(x) = \begin{cases} x^2 - 1, x \le 1 \\ -x^2 - 1, x > 1 \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left[x^2 - 1\right] = 1^2 - 1 = 1 - 1 = 0$

$\lim_{x\to 1'} f(x) = \lim_{x\to 1}\left[-x^2 - 1\right] = -1^2 - 1 = -1 - 1 = -2$

We get $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Answer 25:

The given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$

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$$= \lim_{x \to 0} \left(\frac{-x}{x} \right)$$

$$= \lim_{x \to 0} (-1)$$

$$= -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[\frac{|x|}{x} \right]$$

Apply limits

$$= \lim_{x \to 0} \left[\frac{x}{x} \right]$$

$$= \lim_{x \to 0} (1)$$

$$= 1$$

We get $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer 26:

The given function is

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[\frac{x}{|x|} \right]$$

$$= \lim_{x \to 0} \left[\frac{x}{-x} \right]$$

$$= \lim_{x \to 0} (-1)$$

$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{x}\right]$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

We get $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x\to 5} f(x)$, where $f(x) = |x| - 5$

Answer 27:

Given

$f(X) = |x| - 5$. $\lim_{x\to 5} f(x) = \lim_{x\to 5}[|x| - 5]$

$$= \lim_{x\to 5}(x-5)$$

$$= 5-5$$

$$= 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}(|x| - 5)$$

$$= \lim_{x\to 5}(x-5)$$

$$= 5-5$$

$$= 0$$

$$\therefore \lim_{x\to 5} f(x) = \lim_{x\to 5^+} f(x) = 0$$

Hence, $\lim_{x\to 5} f(x) = 0$

Question 28:

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Suppose $f(x)=\begin{cases} a+bx, & \text{if } x<1 \\ 4, & \text{if } x=0 \\ b-ax, & \text{if } x>1 \end{cases}$

and $\lim_{x\to 1} f(x)=f(1)$ what are possible values of a and b ?

Answer 28:

The given function is

$$f(x)=\begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(a+bx)=a+b$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(b-ax)=b-a$$

$$f(1)=4$$

It is given that $\lim_{x\to 1} f(x)=f(1)$.

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)=f(1)$$

$$\Rightarrow a+b=4 \text{ and } b-a=4$$

We get $a=0$ and $b=4$.

Question 29:

Let $a_1,a_2,\ldots,$ an be fixed real numbers and define a function $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$

What is $\lim_{x\to a_1} f(x)$? For some $a\neq a_1,a_2\ldots a_n$, compute $\lim_{x\to a} f(x)$.

Answer 29:

The aiven function is $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$

$$\lim_{x\to a_1} f(x)=\lim_{x\to a_1}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]$$

$$=\left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]\ldots\left[\lim_{x\to a_1}(x-a_n)\right]$$

simplify

$=(a_1-a_1)(a_1-a_2)\ldots(a_1-a_n)=0$

$\therefore \lim\limits_{x\to 0_1} f(x)=0$

Now, $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]$

$=\left[\lim\limits_{x\to a}(x-a_1)\right]\left[\lim\limits_{x\to a}(x-a_2)\right]\ldots\left[\lim\limits_{x\to a}(x-a_n)\right]$

$=(a-a_1)(a-a_2)\ldots.(a-a_n)$

We get

$\therefore \lim\limits_{x\to a} f(x)=(a-a_1)(a-a_2)\ldots(a-a_n)$

Question 30:

If $f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$

For what value(s) of a does $\lim\limits_{x\to a} f(x)$ exists?

Answer 30:

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(|x|+1)$

$=\lim\limits_{x\to 0}(-x+1)$

$=-0+1$

$=1$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(|x|-1)$

$=\lim_{x\to 0}(x-1)$

$=0-1$

$=-1$

We get

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a<0$,

$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$

$=\lim_{x\to a}(-x+1)$

$=-a+1$

$\lim_{x\to a^+} f(x)=\lim_{x\to a}(|x|+1)$

$=\lim_{x\to a}(-x+1)$

$=-a+1$

$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a} f(x)=-a+1$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When $a>0$

$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|-1)$

$=\lim_{x\to a}(x-1)$

$=a-1$

$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|-1)$

$=\lim_{x\to a}(x-1)$

$=a-1$

$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=a-1$

Thus, limit of $f(x)$ exists at $x=a$, where $a>0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a\neq 0$.

Infinity Learn

Question 31:

If the function $f(x)$ satisfies, $\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$ evaluate $\lim_{x\to 1} f(x)$

Answer 31:

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$$

$$\Rightarrow \lim_{x\to 1}(f(x)-2)$$

$$\Rightarrow \lim_{x\to 1}\left(x^2-1\right)$$

$$\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi\lim_{x\to 1}\left(x^2-1\right)$$

Apply limits

$$\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi\left(1^2-1\right)$$

$$\Rightarrow \lim_{x\to 1}(f(x)-2)=0$$

$$\Rightarrow \lim_{x\to 1} f(x)-\lim_{x\to 1} 2=0$$

$$\Rightarrow \lim_{x\to 1} f(x)-2=0$$

$$\therefore \lim_{x\to 1} f(x)=2$$

Question 32:

If. $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$ For what integers m and n does $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist?

Answer 32:

The given function is

$$f(x)=mx^2+n, x<0$$

$$nx+m, 0\le x\le 1$$

$$nx^3+m, x>1$$

Infinity Learn

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(mx^2+n\right)$$

$$= m(0)^2 + n$$

$$= n$$

$$\lim_{x\to 0^+} f(x) \ = \lim_{x\to 0}(nx+m)$$

$$= n(0)+m$$

$$= m$$

Thus, $\lim_{x\to 0} f(x)$ exists if $m = n$.

$$\lim_{x\to 1^-} f(x) \ = \lim_{x\to 1}(nx+m)$$

$$= n(1)+m$$

$$= m+n$$

$$\lim_{x\to 1^+} f(x) \ = \lim_{x\to 1}\left(nx^3+m\right)$$

$$= n(1)^3 + m$$

$$= m+n$$

$$\therefore \lim_{x\to 1^-} f(x) \ = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$$

Thus $\lim_{x\to 1} f(x)$ exists

Example 5

Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution

We have

$$f'(2) \ = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{3(2+h)-3(2)}{h}$$

$$= \lim_{h\to 0}\frac{6+3h-6}{h} = \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

We get

3 x at $x = 2$ is 3 .

Example 6

Find the derivative of the function $f(x)=2x^2+3x-5$ at $x=-1$. Also prove that $f'(0)+3f'(-1)=0$

Solution

Given

$f(x)$ at $x=-1$ and at $x=0$.

We have

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2(-1+h)^2+3(-1+h)-5\right]-\left[2(-1)^2+3(-1)-5\right]}{h}$$

$$=\lim_{h\to 0}\frac{2h^2-h}{h}$$

$$=\lim_{h\to 0}(2h-1)=2(0)-1=-1 \text{ \\\\ and } \quad f'(0)$$

$$=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

Apply limits

$$=\lim_{h\to 0}\frac{\left[2(0+h)^2+3(0+h)-5\right]-\left[2(0)^2+3(0)-5\right]}{h}$$

$$=\lim_{h\to 0}\frac{2h^2+3h}{h}=\lim_{h\to 0}(2h+3)=2(0)+3=3$$

We get $f'(0)+3f'(-1)=0$

Example 7

Find the derivative of $\sin x$ at $x=0$

Solution

Let $f(x)=\sin x$.

Then

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

Apply limits

$$=\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Example 8

Find the derivative of $f(x)=3$ at $x=0$ and at $x=3$.

Solution

Given that

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{3-3}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Similarly

Apply the limits

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{3-3}{h}=0.$$

We get

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{Q\to P}\frac{QR}{PR}$$

Hence $f'(a)=\tan\psi$

Example 9

Find the derivative of $f(x)=10x$.

Solution

Given

$f(x)=10x$.

Since $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{10(x+h)-10(x)}{h}$$

simplify

$$=\lim_{h\to 0}\frac{10h}{h}=\lim_{h\to 0}(10)=10$$

Example 10

Find the derivative of $f(x)=x^2$.

Solution

Given

$f(x)=x^2$.

We have,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Apply limits

$$=\lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}=\lim_{h\to 0}(h+2x)=2x$$

Example 11

Find the derivative of the constant function $f(x)=a$ for a fixed real number a.

Solution

Given

$f(x)=a$

We have,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Apply limits

$$=\lim_{h\to 0}\frac{a-a}{h}=\lim_{h\to 0}\frac{0}{h}=0 \text{ as } h\neq 0$$

Example 12

Find the derivative of $f(x)=\frac{1}{x}$

Solution

We have $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Infinity Learn

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right]$$

Apply limits

$$= \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Example 13

Compute the derivative of $6x^{100} - x^{55} + x$.

Solution

Given that

$6x^{100} - x^{55} + x$.

differentiate

$$= \frac{d}{dx}\left(6x^{100}\right) - \frac{d}{dx}\left(x^{55}\right) + \frac{d}{dx}(x)$$

$$\frac{d}{dx}\left(6x^{100}\right) = 600x^{99}$$

$$\frac{d}{dx}\left(x^{55}\right) = 55x^{54}$$

$$\frac{d}{dx}(x) = 1$$

$$= 600x^{99} - 55x^{54} + 1$$

Example 14

Find the derivative of $f(x) = 1 + x + x^2 + x^3 + \ldots + x^{50}$ at $x = 1$

Solution

Given that

$f(x) = 1 + x + x^2 + x^3 + \ldots + x^{50}$ at $x = 1$

$1 + 2x + 3x^2 + \ldots + 50x^{49}$.

Infinity Learn

At $x=1$ the value is

$$1+2(1)+3(1)^2+\ldots+50(1)^{49}$$

$$=1+2+3+\ldots+50$$

$$=\frac{(50)(51)}{2}=1275$$

Example 15

Find the derivative of $f(x)=\frac{x+1}{x}$

Solution

Given that

$$f(x)=\frac{x+1}{x}$$

Use $u=x+1$ and $v=x$.

Hence $u'=1$ and $v'=1$.

Therefore

$$\frac{df(x)}{dx}=\frac{d}{dx}\left(\frac{x+1}{x}\right)=\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}=\frac{1(x)-(x+1)1}{x^2}=-\frac{1}{x^2}$$

Example 16

Compute the derivative of $\sin x$.

Solution

Let $f(x)=\sin x$.

Then

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$ (using formula for $\sin A-\sin B$)

Example 17

Compute the derivative of $\tan x$.

Solution

Let $f(x)=\tan x$. Then

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\left[\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}\right]$$

aimplify

$$=\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}$$

$$=1\cdot\frac{1}{\cos^2 x}=\sec^2 x$$

Example 18

Compute the derivative of $f(x)=\sin^2 x$.

Solution

Given

$f(x)=\sin^2 x$.

$$\frac{df(x)}{dx}=\frac{d}{dx}(\sin x\sin x)$$

$$=(\sin x)'\sin x+\sin x(\sin x)'$$

Then

$$=(\cos x)\sin x+\sin x(\cos x)$$

$$=2\sin x\cos x=\sin 2x$$

Exercise 13.2

Question 1:

Find the derivative of x^2-2 at $x=10$.

Answer 1:

Let $f(x)=x^2-2$. Accordingly,

$$f'(10)=\lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$=\lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-\left(10^2-2\right)}{h}$$

$$=\lim_{h\to 0}\frac{10^2+2\cdot 10\cdot h+h^2-2-10^2+2}{h}$$

$$=\lim_{h\to 0}\frac{20h+h^2}{h}$$

$$=\lim_{h\to 0}(20+h)=(20+0)=20$$

We get x^2-2 at $x=10$ is 20

Question 2:

Find the derivative of 99 x at $x=100$.

Answer 2:

Let $f(x)=99x$. Accordingly,

$$=\lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$

$$=\lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$

$$=\lim_{h\to 0}\frac{99h}{h}$$

$$=\lim_{h\to 0}(99)=99$$

We get

99 x at $x=100$ is 99 .

Infinity Learn

Question 3:

Find the derivative of x at $x=1$.

Answer 3:

Let $f(x)=x$. Accordingly,

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)-1}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0}(1)$$

$$=1$$

We get

x at $x=1$ is 1

Question 4:

Find the derivative of the following functions from first principle.

(i) x^3-27

(ii) $(x-1)(x-2)$

(iii) $\frac{1}{x^2}$

(iv) $\frac{x+1}{x-1}$

Answer 4:

(i) Let $f(x)=x^3-27$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$

Infinity Learn

$$= \lim_{h\to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h\to 0} \left(h^2 + 3x^2 + 3xh\right)$$

$$= 0 + 3x^2 + 0 = 3x^2$$

(ii) Let $f(x) = (x-1)(x-2)$.

Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h\to 0} \frac{\left(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2\right) - \left(x^2 - 2x - x + 2\right)}{h}$$

aimplify

$$= \lim_{h\to 0} \frac{\left(hx + hx + h^2 - 2h - h\right)}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h\to 0} (2x + h - 3)$$

$$= (2x + 0 - 3)$$

$$= 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$

We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]$$

$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let $f(x)=\dfrac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\left(x^2+hx+x-x-h-1\right)-\left(x^2+hx-x+x+h-1\right)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Answer 5:

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\ldots+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$,

We get

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\ldots+\frac{2x}{2}+1+0$$

$\therefore f'(x)=x^{99}+x^{98}+\ldots+x+1$ At $x=0$,

$f'(0)=1$ At $x=1$, $f'(1)=1^{99}+1^{98}+\ldots+1+1=[1+1+\ldots+1+1]_{1001\text{tems}}=1\times100=100$

Thus, $f'(1)=100\times f^{1}(0)$

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$ for some fixed real number a.

Answer 6:

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$

$$\therefore f'(x)\ =\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n\right)$$

$$=\frac{d}{dx}\left(x^n\right)+a\frac{d}{dx}\left(x^{n-1}\right)+a^2\frac{d}{dx}\left(x^{n-2}\right)+\ldots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1} + a^n(0)$

$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$

Question 7:

For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $\left(ax^2+b\right)^2$ (iii) $\dfrac{x-a}{x-b}$

Answer 7:

(i) Let $f(x) = (x-a)(x-b)$

$\Rightarrow f(x) = x^2 - (a+b)x + ab$

$\therefore f'(x) = \dfrac{d}{dx}\left(x^2 - (a+b)x + ab\right)$

$= \dfrac{d}{dx}\left(x^2\right) - (a+b)\dfrac{d}{dx}(x) + \dfrac{d}{dx}(ab)$

On using theorem $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$f'(x) = 2x - (a+b) + 0 = 2x - a - b$

(ii) Let $f(x) = \left(ax^2+b\right)^2$

$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$

$\therefore f'(x) = \dfrac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right) = a^2\dfrac{d}{dx}\left(x^4\right) + 2ab\dfrac{d}{dx}\left(x^2\right) + \dfrac{d}{dx}\left(b^2\right)$

On using theorem $\dfrac{d}{dx}x^n = nx^{n-1}$, we obtain

$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$

$= 4a^2x^3 + 4abx$

$= 4ax\left(ax^2+b\right)$

(iii) Let $f(x) = \dfrac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x) \ = \frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.

Answer 8:

Let $f(x) = \frac{x^n - a^n}{x-a}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

By quotient rule,

$$f'(x) \ = \frac{(x-a)\frac{d}{dx}\left(x^n - a^n\right)-\left(x^n - a^n\right)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)\left(nx^{n-1}-0\right)-\left(x^n - a^n\right)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x-\frac{3}{4}$

(ii) $\left(5x^3+3x-1\right)(x-1)$

(iii) $x^{-3}(5+3x)$

(iv) $x^5\left(3-6x^{-9}\right)$

(v) $x^{-4}\left(3-4x^{-5}\right)$

(vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Answer 9:

(i) Let $f(x)=2x-\frac{3}{4}$

$$f'(x) = \frac{d}{dx}\left(2x-\frac{3}{4}\right)$$

$$=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)$$

$$=2-0$$

$$=2$$

(ii) Let $f(x)=\left(5x^3+3x-1\right)(x-1)$

By Leibnitz product rule,

$$f'(x) = \left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right)$$

$$=\left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right)$$

$$=\left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right)$$

$$=5x^3+3x-1+15x^3+3x-15x^2-3$$

$$=20x^3-15x^2+6x-4$$

(iii) Let $f(x)=x^{-3}(5+3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}\left(x^{-3}\right)$$

$$= x^{-3}(0+3)+(5+3x)\left(-3x^{-3-1}\right)$$

$$= x^{-3}(3)+(5+3x)\left(-3x^{-4}\right)$$

$$= 3x^{-3}-15x^{-4}-9x^{-3}$$

$$= -6x^{-3}-15x^{-4}$$

$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$

$$= \frac{-3x^{-3}}{x}(2x+5)$$

$$= \frac{-3}{x^{4}}(5+2x)$$

(iv) Let $f(x)=x^{5}\left(3-6x^{-9}\right)$

By Leibnitz product rule,

$$f'(x) = x^{5}\frac{d}{dx}\left(3-6x^{-9}\right)+\left(3-6x^{-9}\right)\frac{d}{dx}\left(x^{5}\right)$$

$$= x^{5}\left\{0-6(-9)x^{-9-1}\right\}+\left(3-6x^{-9}\right)\left(5x^{4}\right)$$

$$= x^{5}\left(54x^{-10}\right)+15x^{4}-30x^{-5}$$

$$= 54x^{-5}+15x^{4}-30x^{-5}$$

$$= 24x^{-5}+15x^{4}$$

$$= 15x^{4}+\frac{24}{x^{5}}$$

(v) Let $f(x)=x^{-4}\left(3-4x^{-5}\right)$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right)+\left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$

$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+\left(3-4x^{-5}\right)(-4)x^{-4-1}$$

$$= x^{-4}\left(20x^{-6}\right)+\left(3-4x^{-5}\right)\left(-4x^{-5}\right)$$

$$= 20x^{-10}-12x^{-5}+16x^{-10}$$

$$= 36x^{-10}-12x^{-5}$$

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$$=-\frac{12}{x^5}+\frac{36}{x^{10}}$$

(vi) Let $f(x)=\frac{2}{x+1}-\frac{x^2}{3x-1}$

$$f'(x)=\frac{d}{dx}\left(\frac{2}{x+1}\right)-\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x)\ =\left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$=\left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-\left(x^2\right)(3)}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of $\cos x$ from first principle.

Answer 10:

Let $f(x)=\cos x$. Accordingly, from the first principle,

$$f'(x)\ =\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

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$$= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x \sin h}{h}\right]$$

$$= -\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right) - \sin x \lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= -\cos x(0) - \sin x(1)$$

$$= -\sin x' (x) = -\sin x$$

Question 11:

Find the derivative of the following functions:

(i) $\sin x \cos x$

(ii) $\sec x$

(iii) $5\sec x + 4\cos x$

(iv) $\text{cosec}\, x$

(v) $3\cot x + 5\,\text{cosec}\, x$

(vi) $5\sin x - 6\cos x + 7$

(vii) $2\tan x - 7\sec x$

Answer 11:

(i) Let $f(x) = \sin x \cos x$.

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h\to 0}\frac{1}{2h}[2\sin(x+h)\cos(x+h) - 2\sin x \cos x]$$

$$= \lim_{h\to 0}\frac{1}{2h}[\sin 2(x+h) - \sin 2x]$$

$$= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

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$$= \lim_{h \to 0} \frac{1}{h}[\cos(2x+h)\sin h]$$

$$= \lim_{h \to 0} \cos(2x+h) \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos(2x+0) \cdot 1$$

$$= \cos 2x$$

(ii) Let $f(x) = \sec x$.

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \lim_{h \to 0} \frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$= \frac{1}{\cos x} \cdot \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

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$$=\frac{1}{\cos x}\cdot 1\cdot\frac{\sin x}{\cos x}$$

$$=\sec x\tan x$$

(iii) Let $f(x)=5\sec x+4\cos x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)]}{\cos(x+h)}\right]+4\lim\frac{1}{h\to 0}[-\cos x(1-\cosh)-\sin x\sin h]$$

$$=\frac{5}{\cos x}\cdot\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{5}{2}}\right]-4\sin x$$

$$=\frac{5}{\cos x}\cdot\frac{\sin x}{\cos x}\cdot 1-4\sin x$$

$$=5\sec x\tan x\cdot -4\sin x$$

$$=\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x}\cdot\lim_{h\to 0}\left[\frac{\cos(x+h)}{\cos(x+h)}\right]+\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

(iv) Let $f(x)=\operatorname{cosec} x$.

Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec}x]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let $f(x)=3\cot x+5\operatorname{cosec}x$.

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= 3\lim_{h\to 0}\frac{1}{h}[\cot(x+h)-\cot x]+5\lim_{h\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x]$$

Now, $\lim_{h\to 0}\frac{1}{h}[\cot(x+h)-\cot x]$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$= -1\cdot\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x$$

$$\lim_{h\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to \theta}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\frac{(2)}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right)\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec} x\cot x$$

From (1), (2), and (3), we obtain $f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec} x\cot x$

(vi) Let $f(x)=5\sin x-6\cos x+7$.

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7]$$

$$=\lim_{h\to 0}\frac{1}{h}[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}]$$

$$=5\lim_{h\to 0}\frac{1}{h}[\sin(x+h)-\sin x]-6\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{h}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

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$$=5\cos x\cdot 1-6[(-\cos x)\cdot(0)-\sin x\cdot 1]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

(vii) Let $f(x)=2\tan x-7\sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x]$$

$$=\lim_{h\to 0}\frac{1}{h}[2\{\tan(x+h)-\tan x\}-7\{\sec(x+h)-\sec x\}]$$

$$=2\lim_{h\to 0}\frac{1}{h}[\tan(x+h)-\tan x]-7\lim_{h\to 0}\frac{1}{h}[\sec(x+h)-\sec x]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to\frac{h}{2}}\frac{\sin\frac{h}{2}}{h}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1\cdot\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$

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Miscellaneous Examples

Example 19

Find the derivative of f from the first principle, where f is given by

(i) $f(x)=\dfrac{2x+3}{x-2}$

(ii) $f(x)=x+\dfrac{1}{x}$

Solution

(i) we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{2(x+h)+3}{x+h-2}-\frac{2x+3}{x-2}}{h}$$

$$=\lim_{h\to 0}\frac{(2x+2h+3)(x-2)-(2x+3)(x+h-2)}{h(x-2)(x+h-2)}$$

$$=\lim_{h\to 0}\frac{(2x+3)(x-2)+2h(x-2)-(2x+3)(x-2)-h(2x+3)}{h(x-2)(x+h-2)}$$

$$=\lim_{h\to 0}\frac{-7}{(x-2)(x+h-2)}=-\frac{7}{(x-2)^2}$$

f' is not defined at $x=2$.

(ii) The function is not defined at $x=0$. But, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(x+h+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[h+\frac{1}{x+h}-\frac{1}{x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[h+\frac{x-x-h}{x(x+h)}\right]=\lim_{h\to 0}\frac{1}{h}\left[h\left(1-\frac{1}{x(x+h)}\right)\right]$$

$$=\lim_{h\to 0}\left[1-\frac{1}{x(x+h)}\right]=1-\frac{1}{x^2}$$

the function f' is not defined at $x=0$.

Example 20

Find the derivative of $f(x)$ from the first principle, where $f(x)$ is

(i) $\sin x+\cos x$

(ii) $x\sin x$

Solution

(i) we have $f'(x)=\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h+\cos x\cos h-\sin x\sin h-\sin x-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\sin h(\cos x-\sin x)+\sin x(\cos h-1)+\cos x(\cos h-1)}{h}$$
$$=\lim_{h\to 0}\frac{\sin h}{h}(\cos x-\sin x)+\lim_{h\to 0}\sin x\frac{(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{(\cos h-1)}{h}$$
$$=\cos x-\sin x$$

(ii) $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{(x+h)\sin(x+h)-x\sin x}{h}$

$$=\lim_{h\to 0}\frac{(x+h)(\sin x\cos h+\sin h\cos x)-x\sin x}{h}$$
$$=\lim_{h\to 0}\frac{x\sin x(\cos h-1)+x\cos x\sin h+h(\sin x\cos h+\sin h\cos x)}{h}$$
$$=\lim_{h\to 0}\frac{x\sin x(\cos h-1)}{h}+\lim_{h\to 0}x\cos x\frac{\sin h}{h}+\lim_{h\to 0}(\sin x\cos h+\sin h\cos x)$$
$$=x\cos x+\sin x$$

Example 21

Compute derivative of

(i) $f(x)=\sin 2x$

(ii) $g(x)=\cot x$

Solution

(i) We have $\sin 2x=2\sin x\cos x$. Thus $\dfrac{df(x)}{dx}=\dfrac{d}{dx}(2\sin x\cos x)=2\dfrac{d}{dx}(\sin x\cos x)$

$$=2\left[(\sin x)'\cos x+\sin x(\cos x)'\right]$$
$$=2[(\cos x)\cos x+\sin x(-\sin x)]$$

$$= 2\left(\cos^2 x - \sin^2 x\right)$$

(ii) By definition, $g(x) = \cot x = \dfrac{\cos x}{\sin x}$.

then

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{(1)'(\tan x) - (1)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(0)(\tan x) - (\sec x)^2}{(\tan x)^2}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Example 22

Find the derivative of

(i) $\dfrac{x^5 - \cos x}{\sin x}$

(ii) $\dfrac{x + \cos x}{\tan x}$

Solution

(i) Let $h(x) = \dfrac{x^5 - \cos x}{\sin x}$.

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$$h'(x)=\frac{\left(x^5-\cos x\right)'\sin x-\left(x^5-\cos x\right)(\sin x)'}{(\sin x)^2}$$

$$=\frac{\left(5x^4+\sin x\right)\sin x-\left(x^5-\cos x\right)\cos x}{\sin^2 x}$$

$$=\frac{-x^5\cos x+5x^4\sin x+1}{(\sin x)^2}$$

(ii)

We have $\dfrac{x+\cos x}{\tan x}$

$$h'(x)=\frac{(x+\cos x)'\tan x-(x+\cos x)(\tan x)'}{(\tan x)^2}$$

$$=\frac{(1-\sin x)\tan x-(x+\cos x)\sec^2 x}{(\tan x)^2}$$

Miscellaneous Exercise

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x+1)$

(iv) $\cos\left(x-\dfrac{\pi}{8}\right)$

Answer 1:

(i) Let $f(x)=-x$. Accordingly, $f(x+h)=-(x+h)$

We know that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$

$$=\lim_{h\to 0}\frac{-x-h+x}{h}$$

$$=\lim_{h\to 0}\frac{-h}{h}$$

$$=\lim_{h\to 0}(-1)=-1$$

(ii) Let $f(x)=(-x)^{-1}=\frac{1}{-x}=\frac{-1}{x}$. Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

simpify

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{x(x+h)}$$

$$=\frac{1}{x\cdot x}=\frac{1}{x^2}$$

(iii) Let $f(x)=\sin(x+1)$. Accordingly, $f(x+h)=\sin(x+h+1)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[\sin(x+h+1)-\sin(x+1)]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\cos\left(\frac{2x+0+2}{2}\right)\cdot 1$$

$$=\cos(x+1)$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{\left(2x+h-\frac{\pi}{4}\right)\sin\left(\frac{h}{2}\right)}{2}\right)\right]$$

Question 2:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x+a)$

Answer 2:

Given that

$(x+a)$

Let $f(x)=x+a.$

Then, $f(x+h)=x+h+a$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Question 3:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers $(px+q)\left(\frac{r}{x}+s\right)$

Answer 3:

Given

$(px+q)\left(\frac{r}{x}+s\right)$

Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

$$f'(x)=(px+q)\left(\frac{r}{x}+s\right)'+\left(\frac{r}{x}+s\right)(px+q)'$$

$$=(px+q)\left(rx^{-1}+s\right)'+\left(\frac{r}{x}+s\right)(p)$$

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$= (px+q)\left(-rx^{-2}\right)+\left(\frac{r}{x}+s\right)p$

$= (px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$

$= \frac{-pr}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$

$= ps-\frac{qr}{x^2}$

Question 4:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers)

Answer 4:

Let $f(x)=(ax+b)(cx+d)^2$

By product rule,

$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)$

$=(ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)$

$=(ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]$

$=(ax+b)\left(2c^2x+2cd\right)+\left(cx+d^2\right)a$

$=2c(ax+b)(cx+d)+a(cx+d)^2$

Question 5:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Answer 5:

Let

$f(x)=\frac{ax+b}{cx+d}$

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

simplify

$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$=\frac{ad-bc}{(cx+d)^2}$$

Question 6:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer 6:

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1},\ \text{where } x\neq 0$$

By quotient rule,

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},x\neq 0,1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},x\neq 0,1$$

$$=\frac{x-1-x-1}{(x-1)^2},x\neq 0,1$$

$$=\frac{-2}{(x-1)^2},x\neq 0,1$$

Question 7:

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Find the derivative of the following functions (it is to be understood that $a \cdot b, c, d, p, q, r$ and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2+bx+c}$

Answer 7:

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{\left(ax^2+bx+c\right)\frac{d}{dx}(1)-\frac{d}{dx}\left(ax^2+bx+c\right)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

Question 8:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Answer 8:

Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

simplify

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$= \frac{-apx^2 - 2bpx + ar - bq}{\left(px^2 + qx + r\right)^2}$$

Question 9:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2 + qx + r}{ax + b}$

Answer 9:

Let $f(x) = \frac{px^2 + qx + r}{ax + b}$

By quotient rule,

$$f'(x) = \frac{(\alpha x + b)\frac{d}{dx}\left(px^2 + qx + r\right) - \left(px^2 + qx + r\right)\frac{d}{dx}(\alpha x + b)}{(ax + b)^2}$$

$$= \frac{(ax + b)(2px + q) - \left(px^2 + qx + r\right)(a)}{(ax + b)^2}$$

$$= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ar}{(ax + b)^2}$$

$$= \frac{apx^2 + 2bpx + bq - ar}{(ax + b)^2}$$

Question 10:

Find the derivative of the following functions (it is to be understood that $a, b, c, d, p,$ q$, r$ and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Answer 10:

Given

$$\frac{a}{x^4} - \frac{b}{x^2} + \cos x$$

Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x)$$

$$=\frac{-4a}{x^3}+\frac{2b}{x^3}-\sin x$$

Question 11:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $4\sqrt{x}-2$

Answer 11:

Given

$4\sqrt{x}-2$

Let $f(x)=4\sqrt{x}-2$

$$f'(x)=\frac{d}{dx}(4\sqrt{x}-2)=\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that $a,b,c,d,p,$q$,r$ and s are fixed non-zero constants and m and n are integers): $(ax+b)^n$

Answer 12:

Let $f(x)=(ax+b)^n$. Accordingly, $f(x+h)=\{a(x+h)+b\}^n=(ax+ah+b)^n$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{b\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

$$=(ax+b)^n\lim_{b\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{2}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

$$=(ax+b)^n\lim_{b\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{2(ax+b)^2}+\ldots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{(ax+b)}$$

$$=na(ax+b)^{n-1}$$

Question 13:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax+b)^n(cx+d)^m$

Answer 13:

Let $f(x)=(ax+b)^n(cx+d)^m$

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^n\frac{d}{dx}(ax+b)^n$$

Now, let $f_1(x)=(cx+d)^m$

$f_1(x+h)=(cx+ch+d)^{m'}$

$f_1'(x)=\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$

$=\lim_{h\to 0}\frac{(cx+ch+d)^{m'}-(cx+d)^m}{h}$

$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$

$=(cx+d)^m\lim_{b\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{\left(c^2h^2\right)}{(cx+d)^2}+\ldots\right)-1\right]$

$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\ldots(\text{ Terms containing higher degrees of } h)\right]$

$=(cx+d)^m\lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+\ldots\right]$

$=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]$

$=\frac{mc(cx+d)^m}{(cx+d)}$

$=mc(cx+d)^{m-1}$

$\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1}$

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1}$

Therefore, from (1), (2), and (3), we obtain

$f'(x)\ =(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}$

$=(ax+b)^{n-1}(cx+d)^{m-1}[mc(ax+b)+na(cx+d)]$

Question 14:

Find the derivative of the following functions (it is to be understood that a,b,c,d,p,q,r and s are fixed non-zero constants and m and n are integers $)$: $\sin(x+a)$

Answer 14:

Let $f(x)=\sin(x+a)$, therefore $f(x+h)=\sin(x+h+a)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\{\frac{\sin\left(\frac{h}{2}\right)]}{\left(\frac{h}{2}\right)}]\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{b}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

$$=\cos\left(\frac{2x+2a}{2}\right)\times 1$$

$$=\cos(x+a)$$

Question 15:

Find the derivative of the following functions (it is to be understood that a,b,c,d,p,q,r and s are fixed non-zero constants and m and n are integers): $\operatorname{cosec} x\cot x$

Answer 15:

Let $f(x)=\operatorname{cosec} x\cot x$

By product rule,

$$f'(x)=\operatorname{cosec} x(\cot x)'+\cot x(\operatorname{cosec} x)'$$

Let $f_1(x)=\cot x$. Accordingly, $f_1(x+h)=\cot(x+h)$

By first principle,

$$f_1'(x)=\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}\cdot\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=\frac{-1}{\sin x}\cdot 1\cdot\left(\frac{1}{\sin(x+0)}\right)$$

$$=\frac{-1}{\sin^2 x}$$

$$=-\text{cosec}^2 x$$

$$\therefore (\cot x)'=-\text{cosec}^2 x$$

Now, let $f_2(x)=\text{cosec}\, x$. Accordingly, $f_2(x+h)=\text{cosec}(x+h)$ By first principle,

$$=\lim_{h\to 0}\frac{1}{h}[\text{cosec}(x+h)-\text{cosec}\, x]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\cdot\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\cdot\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=\frac{-1}{\sin x}\cdot 1\cdot\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=\frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x}$$

$$=-\operatorname{cosec} x\cdot\cot x$$

$$\therefore(\operatorname{cosec} x)'=-\operatorname{cosec} x\cdot\cot x$$

(3)

$$f'(x)\;=\operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right)+\cot x(-\operatorname{cosec} x\cot x)$$

$$=-\operatorname{cosec}^3 x-\cot^2 x\operatorname{cosec} x$$

Question 16:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1+\sin x}$

Answer 16:

Let $f(x)=\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)}$$

Question 17:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer 17:

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

Question 18:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sec x - 1}{\sec x + 1}$

Answer 18:

Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2\sin x}{(\sec x + 1)^2}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x + 1)^2}$$

$$\frac{2\sin x}{\cos x}\sec x$$

$$= \frac{2\sec x + 1)^2}{(\sec x + 1)^2}$$

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$$= \frac{2\sec x \tan x}{(1+5)^2}$$

Question 19:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Answer 19:

Let $y = \sin^n x$.

Accordingly, for $n = 1, y = \sin x$.

$\therefore \frac{dy}{dx} = \cos x$, i.e., $\frac{d}{dx}\sin x = \cos x$

For $n = 2, y = \sin^2 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x(\sin x)'$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2\sin x \cos x$$

For $n = 3, y = \sin^3 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sin x \sin^2 x\right)$$

$$= (\sin x)' \sin^2 x + \sin x\left(\sin^2 x\right)'$$

$$= \cos x \sin^2 x + \sin x(2\sin x \cos x)$$

$$= \cos x \sin^2 x + 2\sin^2 x \cos x$$

$$= 3\sin^2 x \cos x$$

We have

$$\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$$

It is truefor $n = k$.

i.e., $\frac{d}{dx}\left(\sin^k x\right) = k\sin^{(k-1)} x \cos x$

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$\ldots(2)$

Consider

$$\frac{d}{dx}\left(\sin^{k+1} x\right)=\frac{d}{dx}\left(\sin x \sin^{k} x\right)$$

$$=(\sin x)' \sin^{k} x+\sin x\left(\sin^{k} x\right)'$$

$$=\cos x \sin^{k} x+\sin x\left(k \sin^{(t-1)} x \cos x\right)$$

$$=\cos x \sin^{k} x+k \sin^{k} x \cos x$$

$$=(k+1) \sin^{k} x \cos x$$

Question 20:

Find the derivative of the following functions (it is to be understood that $a, b, c, d, p, \mathrm{q}, r$ and s are fixed non-zero constants and m and n are integers): $\frac{a+b \sin x}{c+d \cos x}$

Answer 20:

Let $f(x)=\frac{a+b \sin x}{c+d \cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d \cos x) \frac{d}{dx}(a+b \sin x)-(a+b \sin x) \frac{d}{dx}(c+d \cos x)}{(c+d \cos x)^{2}}$$

$$=\frac{(c+d \cos x)(b \cos x)-(a+b \sin x)(-d \sin x)}{(c+d \cos x)^{2}}$$

$$=\frac{cb \cos x+bd \cos^{2} x+ad \sin x+bd \sin^{2} x}{(c+d \cos x)^{2}}$$

$$=\frac{bc \cos x+ad \sin x+bd\left(\cos^{2} x+\sin^{2} x\right)}{(c+d \cos x)^{2}}$$

$$=\frac{bc \cos x+ad \sin x+bd}{(c+d \cos x)^{2}}$$

Question 21:

Find the derivative of the following functions (it is to be understood that $a, b, c, d, p, \mathrm{q}, r$ and s are fixed non-zero constants and m and n are integers $)$:

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Answer 21:

$$\frac{\sin(x+a)}{\cos x}$$

Let $f(x)=\dfrac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\dfrac{d}{dx}[\sin(x+a)]-\sin(x+a)\dfrac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\dfrac{d}{dx}[\sin(x+a)]-\sin(x+a)(-\sin x)}{\cos^2 x}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[\sin(x+h+a)-\sin(x+a)]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}]\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

$$=\left(\cos\frac{2x+2a}{2}\right)\times 1$$

$$=\cos(x+a)$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

Question 22:

Find the derivative of the following functions (it is to be understood that a,b,c,d,p,q,r and s are fixed non-zero constants and m and n are integers): $x^4(5\sin x-3\cos x)$

Answer 22:

Let $f(x)=x^4(5\sin x-3\cos x)$

By product rule,

$$f'(x)=x^4\frac{d}{dx}(5\sin x-3\cos x)+(5\sin x-3\cos x)\frac{d}{dx}\left(x^4\right)$$

$$=x^4\left[5\frac{d}{dx}(\sin x)-3\frac{d}{dx}(\cos x)\right]+(5\sin x-3\cos x)\frac{d}{dx}\left(x^4\right)$$

$$=x^4[5\cos x-3(-\sin x)]+(5\sin x-3\cos x)\left(4x^3\right)$$

$$=x^3[5x\cos x+3x\sin x+20\sin x-12\cos x]$$

Question 23:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers $\left(x^2+1\right)\cos x$

Answer 23:

Given

$\left(x^2+1\right)\cos x$

Let $f(x)=\left(x^2+1\right)\cos x$

By product rule,

$$f'(x)=\left(x^2+1\right)\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}\left(x^2+1\right)$$

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$$=\left(x^2+1\right)(-\sin x)+\cos x(2x)$$

$$=-x^2\sin x-\sin x+2x\cos x$$

Question 24:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q,r and s are fixed non-zero constants and m and n are integers): $\left(ax^2+\sin x\right)(p+q\cos x)$

Answer 24:

Let $f(x)=\left(ax^2+\sin x\right)(p+q\cos x)$

By product rule,

$$f'(x)=\left(ax^2+\sin x\right)\frac{d}{dx}(p+q\cos x)+(p+q\cos x)\frac{d}{dx}\left(ax^2+\sin x\right)$$

$$=\left(ax^2+\sin x\right)(-q\sin x)+(p+q\cos x)(2ax+\cos x)$$

$$=-q\sin x\left(ax^2+\sin x\right)+(p+q\cos x)(2ax+\cos x)$$

Question 25:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers) $:(x+\cos x)(x-\tan x)$

Answer 25:

Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x)$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\cdot\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

$$=\frac{1}{\cos x}\cdot 1\cdot\frac{1}{\cos(x+0)}$$

$$=\sec^2 x$$

Therefore, from (i) and (ii), we obtain

$$f'(x)=(x+\cos x)\left(1-\sec^2 x\right)+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)\left(-\tan^2 x\right)+(x-\tan x)(1-\sin x)$$

$$=-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$$

Question 26:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer 26:

Let $f(x)=\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer 27:

Let $f(x)=\dfrac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\frac{\pi}{4}\cdot\left[\frac{\sin x\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$=\cos\frac{\pi}{4}\cdot\left[\frac{\sin x\cdot 2x-x^2\cos x}{\sin^2 x}\right]$$

$$=\frac{x\cos\frac{\pi}{4}[2\sin x-x\cos x]}{\sin^2 x}$$

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Question 28:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+\tan x}$

Answer 28;

Let $f(x)=\frac{x}{1+\tan x}$

$$f'(x)=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x)=\frac{(1+\tan x)-x\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan(x+h)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{b\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow\frac{d}{dx}(1+\tan x)=\sec^2 x$$

From (i) and (ii), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Question 29:

Find the derivative of the following functions (it is to be understood thata, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers $(x+\sec x)(x-\tan x)$

Answer 29:

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right]$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)\cos x} \right]$$

$$= \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x} \right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

$$\tan x = \sec^2 x$$

$$\left. \begin{matrix} dx \\ = \quad h \end{matrix} \right]$$

$$f_2'(x) = \lim_{h \to 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sec(x+h) - \sec x}{h} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right]$$

$$= \sec x \cdot \frac{\left\{\lim_{h\to 0} \sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

$$= \sec x, \frac{\sin x, 1}{\cos x}$$

$$\Rightarrow \frac{d}{dx}\sec x = \sec x \tan x$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)\left(1 - \sec^2 x\right) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Answer 30:

Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

We get

$$\frac{d}{dx}\sin^n x = n \sin^{n-1} x \cos x$$

Therefore,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

$$= \frac{\sin^n x.1 - x\left(n \sin^{n-1} x \cos x\right)}{\sin^{2n} x}$$

$$= \frac{\sin^{n-1} x(\sin x - nx \cos x)}{\sin^{2n} x}$$

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$$= \frac{\sin x - nx\cos x}{\sin^{n+1} x}$$

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