Chapter 2: Relations and Functions

Example 1: If $(x+1, y-2)=(3,1)$, find the values of x and y.

Solution: Given that, $(x+1, y-2)=(3,1)$

Here, the ordered pairs are equal

So, the corresponding elements are equal.

That is, $x+1=3$ and $y-2=1$.

Consider $x+1=3$.

Subtract 1 from both sides,

$x=2$

Now, we have $y-2=1$.

Add 2 to both sides,

$y=3$

Therefore, the values of x and y are 2 and 3 respectively.

Example 2: If $\mathrm{P}=\{a,b,c\}$ and $\mathrm{Q}=\{r\}$, form the sets $\mathrm{P}\times\mathrm{Q}$ and $\mathrm{Q}\times\mathrm{P}$. Are these two products equal?

Solution: Given that, $\mathrm{P}=\{a,b,c\}$ and $\mathrm{Q}=\{r\}$

The Cartesian product of two sets X and Y, denoted $\mathrm{X}\times\mathrm{Y}$, is the set of all ordered pairs where x is in X and y is in Y.

First form the sets $\mathrm{P}\times\mathrm{Q}$ and $\mathrm{Q}\times\mathrm{P}$:

By the definition of the cartesian product,

$\mathrm{P}\times\mathrm{Q}=\{(a,r),(b,r),(c,r)\}$ and $\mathrm{Q}\times\mathrm{P}=\{(r,a),(r,b),(r,c)\}$

Since, by the definition of equality of ordered pairs, the pair (a,r) is not equal to the pair (r,a).

Therefore, $\mathrm{P}\times\mathrm{Q}\neq\mathrm{Q}\times\mathrm{P}$.

But, the number of elements in each set will be the same.

Example 3: Let $\mathrm{A}=\{1,2,3\}, \mathrm{B}=\{3,4\}$ and $\mathrm{C}=\{4,5,6\}$. Find

(i) $\mathrm{A}\times(\mathrm{B}\cap\mathrm{C})$

(ii) $(A\times B)\cap(A\times C)$

(iii) $A\times(B\cup C)$

(iv) $(A\times B)\cup(A\times C)$

Solution: Given that, $A=\{1,2,3\}, B=\{3,4\}$ and $C=\{4,5,6\}$.

The *intersection of two* sets is the *collection* of elements that are common to each of the given *sets*.

The Cartesian product of two sets X and Y, denoted $X\times Y$, is the set of all ordered pairs where x is in X and y is in Y.

The union of two sets is the set of all different elements that are included in either of the two sets.

(i) By the definition of the intersection of two sets, $(B\cap C)=\{4\}$.

Therefore, $A\times(B\cap C)=\{(1,4),(2,4),(3,4)\}$

(ii) Now $(A\times B)=\{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$ and
$(A\times C)=\{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$

Therefore, $(A\times B)\cap(A\times C)=\{(1,4),(2,4),(3,4)\}$

(iii) Since, $(B\cup C)=\{3,4,5,6\}$, we have

$A\times(B\cup C)=\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

(iv) $(A\times B)=\{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$ and
$(A\times C)=\{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$

Then,
$(A\times B)\cup(A\times C)=\{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

Example 4: If $P=\{1,2\}$, form the set $P\times P\times P$.

Solution: Given that, $P=\{1,2\}$

The Cartesian product of two sets X and Y, denoted $X\times Y$, is the set of all ordered pairs where x is in X and y is in Y.

Then, $P\times P\times P=\{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\}$,

Example 5: If $\mathbf{R}$ is the set of all real numbers, what do the cartesian products $\mathbf{R}\times\mathbf{R}$ and $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ represent?

Solution: The Cartesian product $\mathbf{R}\times\mathbf{R}$ represents the set $\mathbf{R}\times\mathbf{R}=\{(x,y):x,y\in\mathbf{R}\}$ which represents the coordinates of all the points in two-dimensional space.

Similarly, the cartesian product $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ represents the set $\mathbf{R}\times\mathbf{R}\times\mathbf{R}=\{(x,y,z):x,y,z\in\mathbf{R}\}$ which represents the coordinates of all the points in three-dimensional space.

Example 6: If $\mathrm{A}\times\mathrm{B}=\{(p,q),(p,r),(m,q),(m,r)\}$, find A and B.

Solution: Given that, $\mathrm{A}\times\mathrm{B}=\{(p,q),(p,r),(m,q),(m,r)\}$

$\mathrm{A}=$ set of first elements

That is, $\mathrm{A}=\{p,m\}$

$\mathrm{B}=$ set of second elements

That is, $\mathrm{B}=\{q,r\}$

Example 7: Let $\mathrm{A}=\{1,2,3,4,5,6\}$. Define a relation R from A to A by

$\mathrm{R}=\{(x,y):y=x+1\}$

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

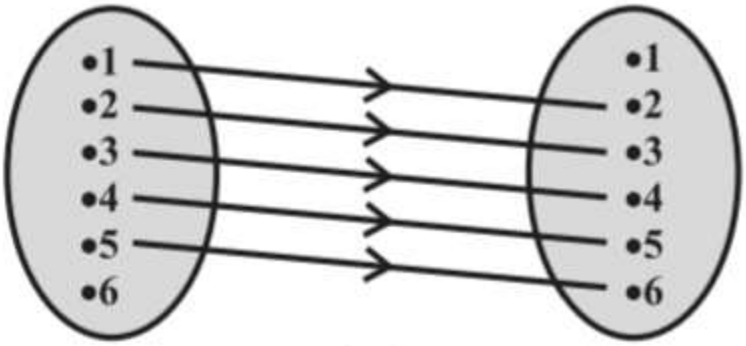
Solution: Given that, $\mathrm{A}=\{1,2,3,4,5,6\}$.

(i) A *relation* can be defined as the *relationship* among sets of values of ordered pairs.

By the definition of the relation,

$\mathrm{R}=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

Now, the corresponding arrow diagram is,



(ii) Here, Domain $=\{1,2,3,4,5\}$,

Similarly, Range $=\{2,3,4,5,6\}$ and the Codomain $=\{1,2,3,4,5,6\}$.

Example 8: The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

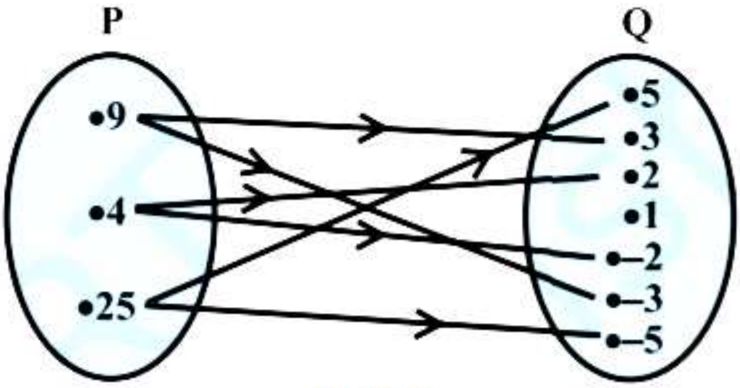


Fig 2.6

Solution : It is observed that the relation R is " x is the square of y ".

(i) In set-builder form, $\mathrm{R}=\{(x, y): x$ is the square of $y, x \in \mathrm{P}, y \in \mathrm{Q}\}$

(ii) In roster form, $\mathrm{R}=\{(9,3),(9,-3),(4,2),(4,-2),(25,5),(25,-5)\}$

The domain of this relation is $\{4,9,25\}$.

The range of this relation is $\{-2,2,-3,3,-5,5\}$.

We can observe that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

Example 9 : Let $\mathrm{A}=\{1,2\}$ and $\mathrm{B}=\{3,4\}$. Find the number of relations from A to B.

Solution : Given that, $\mathrm{A}=\{1,2\}$ and $\mathrm{B}=\{3,4\}$.

Then, $\mathrm{A} \times \mathrm{B}=\{(1,3),(1,4),(2,3),(2,4)\}$

Here, $n(\mathrm{A} \times \mathrm{B})=4$

So, the number of subsets of $\mathrm{A} \times \mathrm{B}$ is 2^{4}.

Therefore, the number of relations from A into B will be 2^{4}.

Example 10 : Let $\mathbf{N}$ be the set of natural numbers and the relation R be defined on N such that $\mathrm{R}=\{(x, y): y=2x, x, y \in \mathbf{N}\}$. What is the domain, codomain and range of R ? Is this relation a function?

Solution : Given that, $\mathrm{R}=\{(x, y): y=2x, x, y \in \mathbf{N}\}$.

The domain of R is the set of natural numbers $\mathbf{N}$.

The codomain is also the set of natural numbers $\mathbf{N}$.

The range is the set of even natural numbers.

This relation is a function. Because, every natural number n has one and only one image.

Example 11 : Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) $\mathrm{R}=\{(2,1),(3,1),(4,2)\}$

(ii) $\mathrm{R}=\{(2,2),(2,4),(3,3),(4,4)\}$

(iii) $\mathrm{R}=\{(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)\}$

Solution :

(i) The relation $\mathrm{R}=\{(2,1),(3,1),(4,2)\}$ is a function. Because, $2,3,4$ are the elements of domain of R having their unique images.

(ii) The relation $\mathrm{R}=\{(2,2),(2,4),(3,3),(4,4)\}$ is not a function. Because, the same first element 2 corresponds to two different images 2 and 4.

(iii) The relation $\mathrm{R}=\{(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)\}$ is a function. Because, every element has one and only one image.

Example 12 : Let $\mathbf{N}$ be the set of natural numbers. Define a real valued function $f:\mathbf{N}\to\mathbf{N}$ by $f(x)=2x+1$. Using this definition, complete the table given below.

x	1	2	3	4	5	6	7
y	$f(1)=...$	$f(2)=...$	$f(3)=...$	$f(4)=...$	$f(5)=...$	$f(6)=...$	$f(7)=...$

Solution : Calculate the values of y by substituting each value of x in $f(x)=2x+1$.

The completed table is,

x	1	2	3	4	5	6	7
y	$f(1)=3$	$f(2)=5$	$f(3)=7$	$f(4)=9$	$f(5)=11$	$f(6)=13$	$f(7)=15$

Example 13 : Define the function $f:\mathbf{R}\to\mathbf{R}$ by $y=f(x)=x^2, x\in\mathbf{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y=f(x)=x^2$									

Solution : Calculate the values of y by substituting each value of x in $f(x) = 2x+1$.

The completed table is,

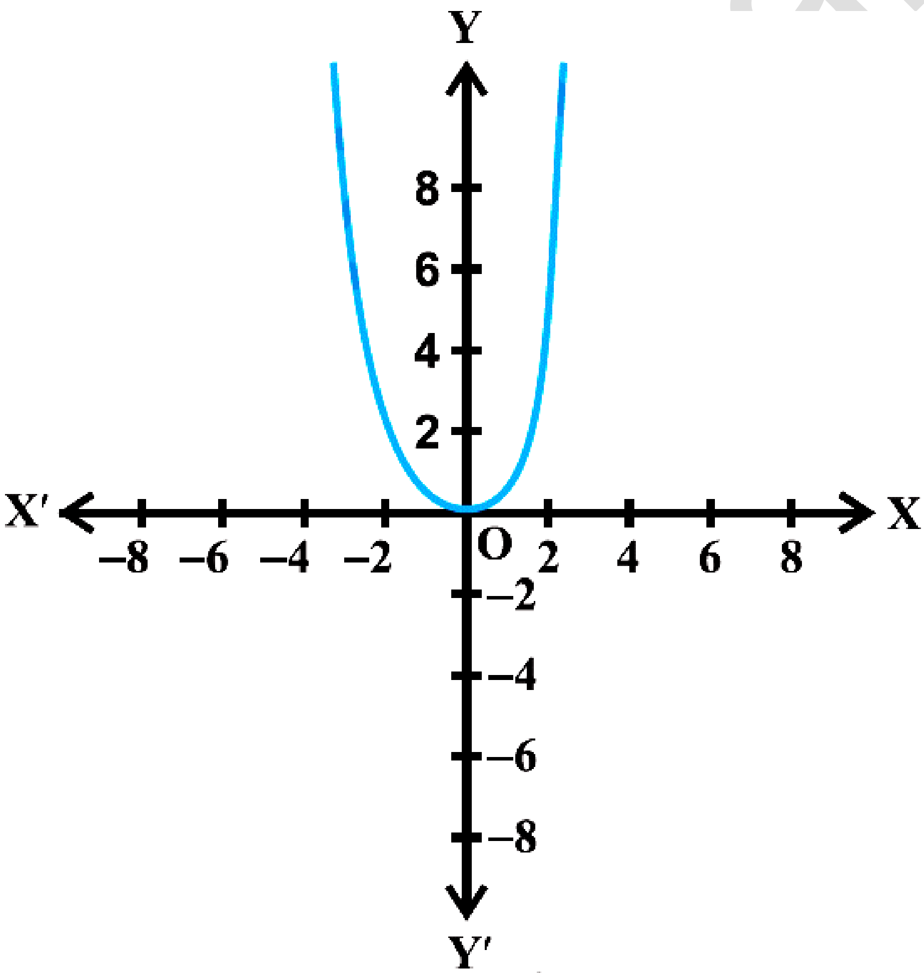
x	–4	–3	–2	–1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

The domain is the set of possible input values and the range is the set of possible output values.

Domain of $f = \{x : x \in \mathbf{R}\}$.

Range of $f = \{x^2 : x \in \mathbf{R}\}$.

The graph of f is,



$f(x) = x^2$

Example 14 : Draw the graph of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3, x \in \mathbf{R}$.

Solution : Given that, $f(x) = x^3, x \in \mathbf{R}$.

Then, $f(0) = 0$

$f(1) = 1$

$f(-1) = -1$

$f(2) = 8$

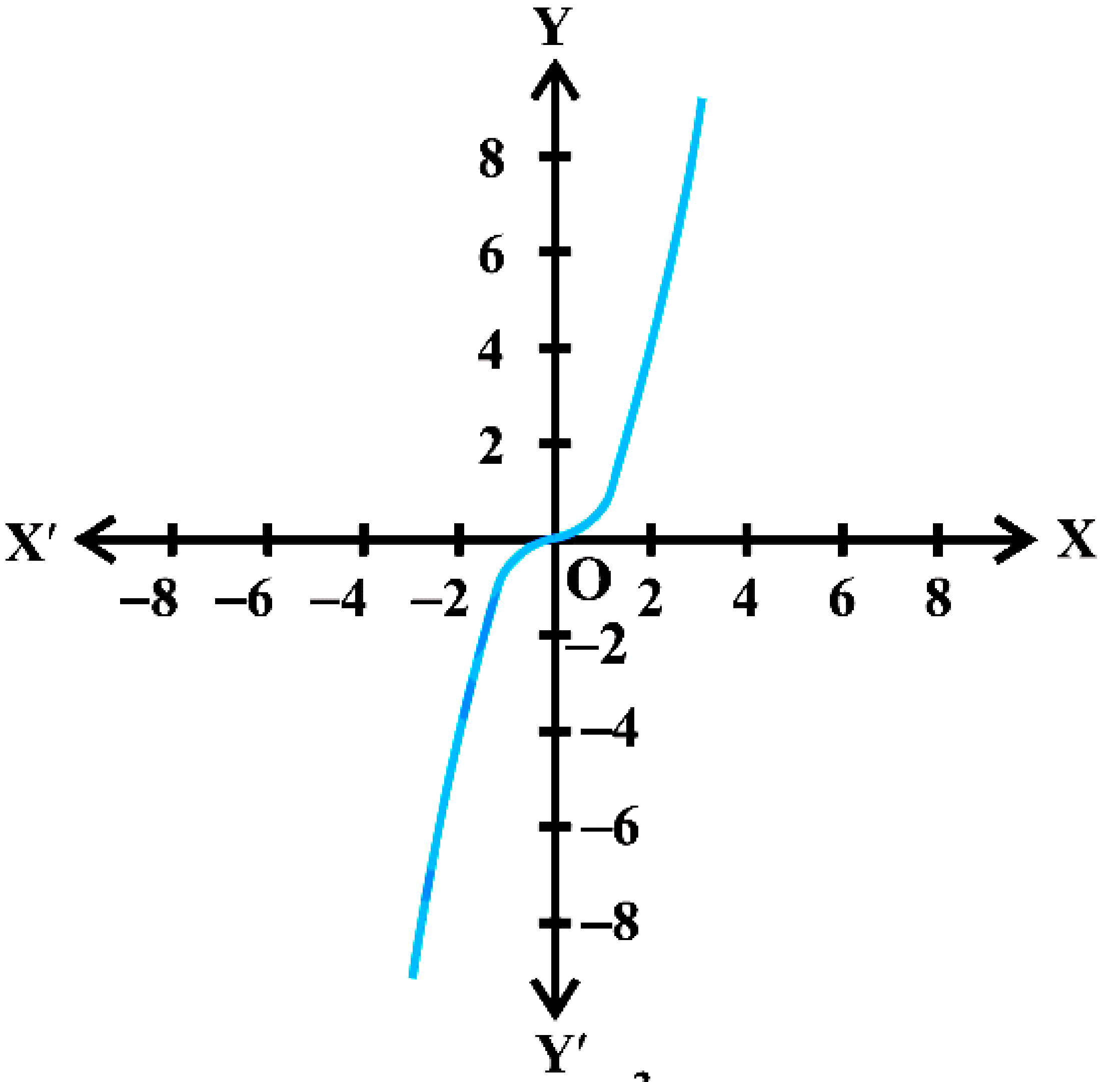
$f(-2) = -8$

$f(3) = 27$

$f(-3) = -27$

Therefore, $f = \left\{\left(x, x^3\right) : x \in \mathbf{R}\right\}$.

The graph of f is,



$f(x) = x^3$

Example 15 : Define the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) = \frac{1}{x}$ $x \in \mathbf{R} - \{0\}$.

Complete the Table given below using this definition. What is the domain and range of this function?

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	…	…	…	…	…	…	…		

Solution : Calculate the values of y by substituting each value of x in $f(x) = 2x + 1$.

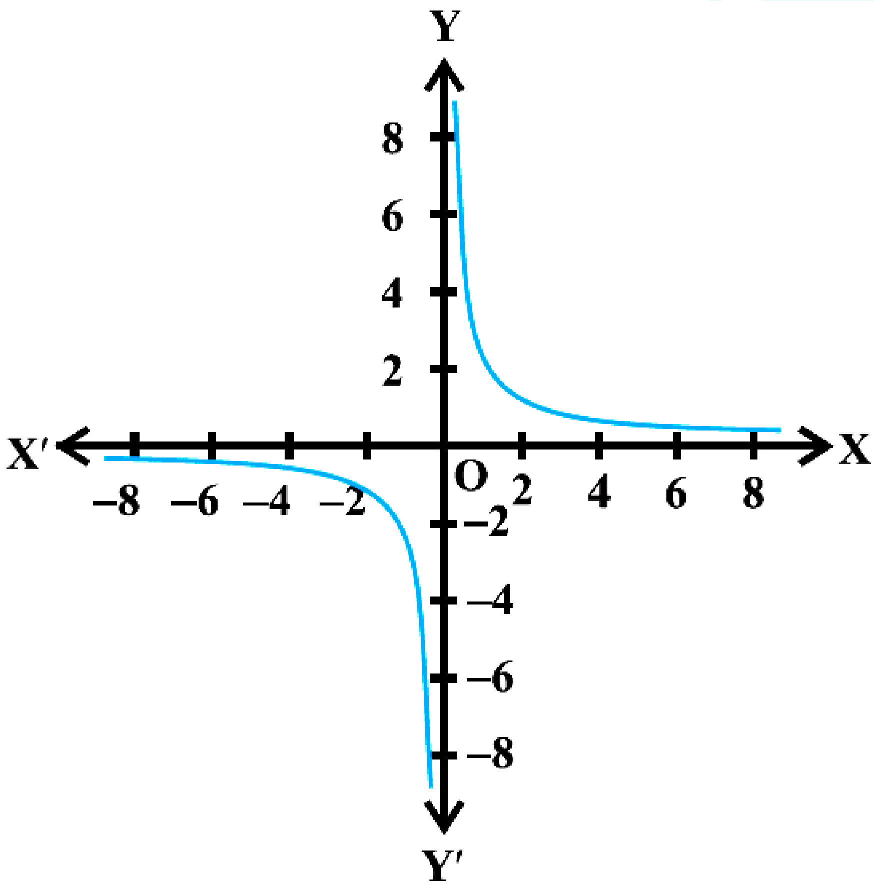
The completed table is,

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2

$y=\frac{1}{x}$	-0.5	-0.67	-1	-2	4	2	1	0.67	0.5

The domain is all real numbers except 0 and its range is also all real numbers except 0.

The graph of f is,



$f(x) = \frac{1}{x}$

Example 16 : Let $f(x)=x^2$ and $g(x)=2x+1$ be two real functions. Find

$(f+g)(x), (f-g)(x), (fg)(x), \left(\frac{f}{g}\right)(x)$.

Solution : Given that, $f(x)=x^2$ and $g(x)=2x+1$.

Then, $(f+g)(x)=x^2+2x+1$

$(f-g)(x)=x^2-2x-1$

$(fg)(x) = x^2(2x+1) = 2x^3 + x^2$

$$\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}, x \neq -\frac{1}{2}$$

Example 17 : Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x), (f-g)(x), (fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution : Given that, $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers.

Then, $(f+g)(x) = \sqrt{x} + x$

$(f-g)(x) = \sqrt{x} - x$

$(fg)x = \sqrt{x}(x) = x^{\frac{3}{2}}$

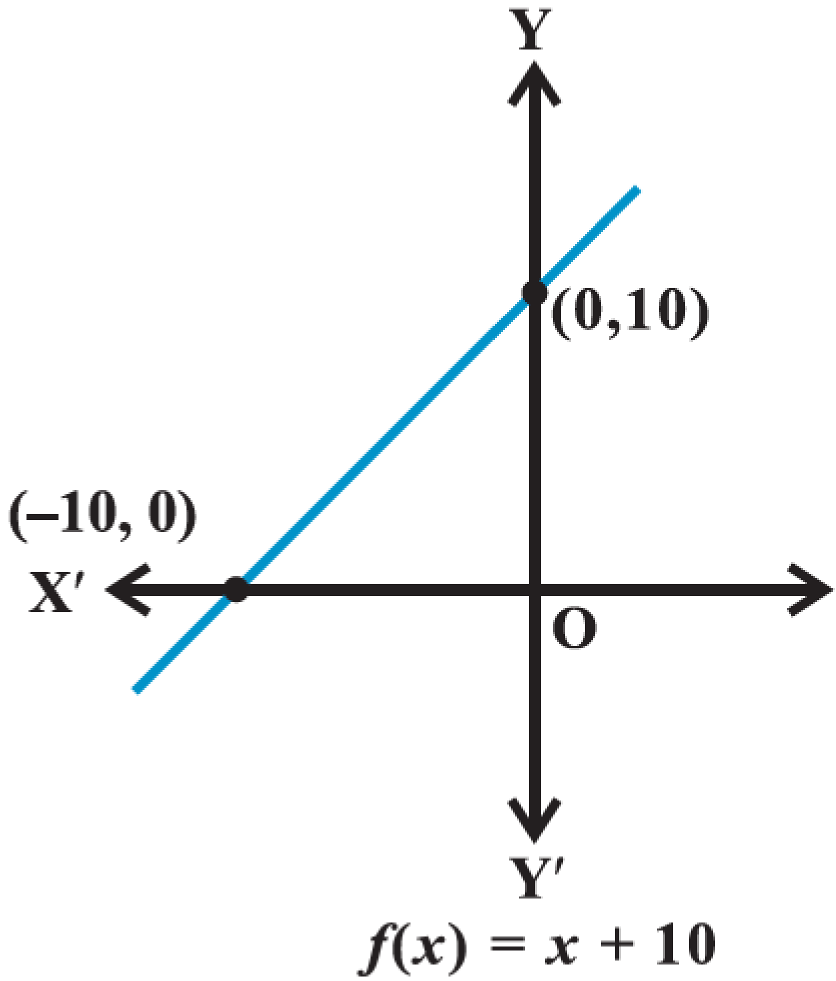
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}}, x \neq 0$$

Example 18 : Let $\mathbf{R}$ be the set of real numbers. Define the real function $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x + 10$ and sketch the graph of this function.

Solution : Here, $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x + 10$

Then, $f(0) = 10, f(1) = 11, f(2) = 12, \ldots \; f(10) = 20$, etc., and $f(-1) = 9, f(-2) = 8, \ldots, f(-10) = 0$ and so on.

The graph of the given function is,



Example 19 : Let R be a relation from $\mathbf{Q}$ to $\mathbf{Q}$ defined by $R=\{(a,b): a,b \in \mathbf{Q}$ and $a-b \in \mathbf{Z}\}$. Show that

(i) $(a,a) \in R$ for all $a \in \mathbf{Q}$

(ii) $(a,b) \in R$ implies that $(b,a) \in R$

(iii) $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$

Solution : Given that, $R=\{(a,b): a,b \in \mathbf{Q}$ and $a-b \in \mathbf{Z}\}$.

(i) Since, $a-a=0 \in \mathbf{Z}$, it follows that $(a,a) \in R$.

Therefore, $(a,a) \in R$ for all $a \in \mathbf{Q}$.

(ii) $(a,b) \in R$ implies that $a-b \in \mathbf{Z}$.

Then, $b-a \in \mathbf{Z}$.

Thus, $(b,a) \in R$

(iii) (a,b) and $(b,c)\in \mathrm{R}$ implies that $a-b\in \mathbf{Z}.b-c\in \mathbf{Z}$.

Then, $a-c=(a-b)+(b-c)\in \mathbf{Z}.$

Thus, $(a,c)\in \mathrm{R}$

Example 20 : Let $f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a linear function from $\mathbf{Z}$ into $\mathbf{Z}$. Find $f(x)$.

Solution : Here, f is a linear function.

Then, $f(x)=mx+c.$

Also, since $(1,1),(0,-1)\in \mathrm{R}$, $f(1)=m+c=1$ and $f(0)=c=-1.$

Then, $m=2$ and $f(x)=2x-1$.

Example 21 : Find the domain of the function $f(x)=\dfrac{x^2+3x+5}{x^2-5x+4}$.

Solution : Given that, $f(x)=\dfrac{x^2+3x+5}{x^2-5x+4}$.

We know, $x^2-5x+4=(x-4)(x-1)$.

Then, the function f is defined for all real numbers except at $x=4$ and $x=1$.

Thus, the domain of f is $\mathbf{R}-\{1,4\}$.

Example 22 : The function f is defined by $f(x)=\begin{cases}1-x, & x<0\\ 1, & x=0\\ x+1, & x>0\end{cases}$

Draw the graph of $f(x)$.

Solution : Here, $f(x)=1-x, x<0$.

Then,

$f(-4)=1-(-4)=5$

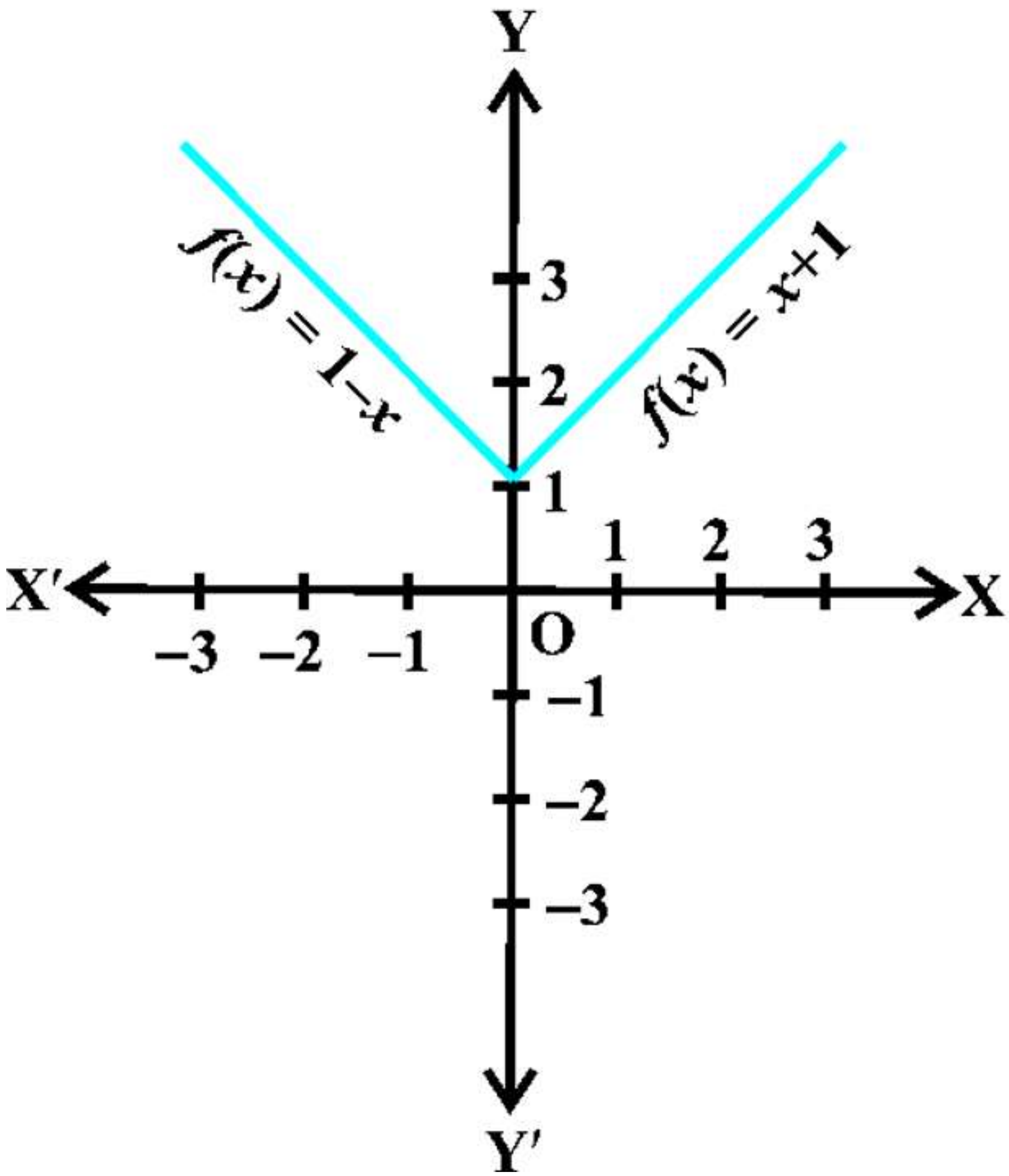
$f(-3)=1-(-3)=4$

$f(-2)=1-(-2)=3$

$f(-1)=1-(-1)=2;$ etc,

and $f(1)=2,\ f(2)=3,\ f(3)=4\ f(4)=5$ and so on for $f(x)=x+1, x>0$

Therefore, the graph of f is,



Exercise 2.1

Question 1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$, find the values of x and y.

Solution : It is given that,

$\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$

Since the ordered pairs are equal, the corresponding elements will also be equal.

Thus, $\frac{x}{3}+1=\frac{5}{3}$ and $y-\frac{2}{3}=\frac{1}{3}$

Consider $\frac{x}{3}+1=\frac{5}{3}$,

Subtract 1 from both sides,

$\frac{x}{3}=\frac{5}{3}-1$

Subtract the numbers,

$\frac{x}{3}=\frac{2}{3}$

Multiple both sides by 3,

$x=2$

Now, $y-\frac{2}{3}=\frac{1}{3}$

Add $\frac{2}{3}$ to both sides,

$y=\frac{1}{3}+\frac{2}{3}$

Add the numbers,

$y=1$

Therefore, $x=2$ and $y=1$.

Question 2. If the set A has 3 elements and the set $\mathrm{B}=\{3,4,5\}$, then find the number of elements in $(\mathrm{A}\times\mathrm{B})$?

Solution : It is given that set A has 3 elements and $\mathrm{B}=\{3,4,5\}$

Number of elements in set $\mathrm{B}=3$

Number of elements in $(\mathrm{A}\times\mathrm{B})=($ Number of elements in $\mathrm{A})\times($ Number of elements in $\mathrm{B})$

$=3\times3=9$

Therefore, the number of elements in $(\mathrm{A}\times\mathrm{B})$ is 9 .

Question 3. If $G=\{7,8\}$ and $H=\{5,4,2\}$, find $G\times H$ and $H\times G$.

Solution : Given that, $G=\{7,8\}$ and $H=\{5,4,2\}$

It is known that the Cartesian product $P\times Q$ of two non-empty sets P and Q is defined as $P\times Q=\{(p,q): p\in P, q\in Q\}$.

Therefore, $G\times H=\{(7,5),(7,4),(7,2),(8,5),(8,4),(8,2)\}$

$H\times G=\{(5,7),(5,8),(4,7),(4,8),(2,7),(2,8)\}$

Question 4. State whether each of the following statement are true or false. If the statement is false, rewrite the given statement correctly.

(i) If $P=\{m,n\}$ and $Q=\{n,m\}$, then $P\times Q=\{(m,n),(n,m)\}$.

(ii) If A and B are non-empty sets, then $A\times B$ is a non-empty set of ordered pairs (x,y) such that $x\in A$ and $y\in B$.

(iii) If $A=\{1,2\}, B=\{3,4\}$, then $A\times(B\cap\Phi)=\Phi$.

Solution :

(i) The Cartesian product of two sets X and Y, denoted $X\times Y$, is the set of all ordered pairs where x is in X and y is in Y.

If $P=\{m,n\}$ and $Q=\{n,m\}$, then $P\times Q=\{(m,m),(m,n),(n,m),(n,n)\}$.

So, the statement is false.

(ii) The Cartesian product of two sets X and Y, denoted $X\times Y$, is the set of all ordered pairs where x is in X and y is in Y.

If A and B are non-empty sets, then $A\times B$ is a non-empty set of ordered pairs (x,y) such that $x\in A$ and $y\in B$.

So, the statement is true.

(iii) $A=\{1,2\}, B=\{3,4\}$

Then, $A\times(B\cap\Phi)=\Phi$.

So, the statement is true.

Question 5. If $A=\{-1,1\}$, find $A\times A\times A$.

Solution : It is given that $A=\{-1,1\}$.

It is known that for any non-empty set A, $A\times A\times A$ is defined as $A\times A\times A=\{(a,b,c): a,b,c\in A\}$

Therefore,

$$A\times A\times A=\{(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1),(1,-1,-1),(1,-1,1),(1,1,-1),(1,1,1)\}$$

Question 6. If $A\times B=\{(a,x),(a,y),(b,x),(b,y)\}$. Find A and B.

Solution : It is given that $A\times B=\{(a,x),(a,y),(b,x),(b,y)\}$.

It is known that the Cartesian product of two non-empty sets P and Q is defined as

$P\times Q=\{(p,q): p\in P, q\in Q\}$

Thus, A is the set of all first elements and B is the set of all second elements.

Therefore, $A=\{a,b\}$ and $B=\{x,y\}$.

Question 7. Let $A=\{1,2\}, B=\{1,2,3,4\}, C=\{5,6\}$ and $D=\{5,6,7,8\}$. Verify that

(i) $A\times(B\cap C)=(A\times B)\cap(A\times C)$

(ii) $A\times C$ is a subset of $B\times D$

Solution :

(i) To verify: $A\times(B\cap C)=(A\times B)\cap(A\times C)$

It is given that, $B\cap C=\{1,2,3,4\}\cap\{5,6\}=\Phi$

Thus, L.H.S $=A\times(B\cap C)=A\times\Phi=\Phi$

$A\times B=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4)\}$

$A\times C=\{(1,5),(1,6),(2,5),(2,6)\}$

Therefore, R.H.S $=(A\times B)\cap(A\times C)=\Phi$

Thus, L.H.S $=$ R.H.S

Hence, $A\times(B\cap C)=(A\times B)\cap(A\times C)$

(ii) To verify : $A\times C$ is a subset of $B\times D$

Given that, $A=\{1,2\}, B=\{1,2,3,4\}, C=\{5,6\}$ and $D=\{5,6,7,8\}$.

Then, $A\times C=\{(1,5),(1,6),(2,5),(2,6)\}$

$$A\times D=\{(1,5),(1,6),(1,7),(1,8),(2,5),(2,6),(2,7),(2,8),(3,5),(3,6),(3,7),(3,8),(4,5),(4,6),(4,7),(4,8)\}$$

It is seen that all the elements of set $A\times C$ are the elements of set $B\times D$.

Therefore, $A\times C$ is a subset of $B\times D$.

Question 8. Let $A=\{1,2\}$ and $B=\{3,4\}$. Write $A\times B$. How many subsets will $A\times B$ have? List them.

Solution : $A=\{1,2\}$ and $B=\{3,4\}$

Thus, $A\times B=\{(1,3),(1,4),(2,3),(2,4)\}$

$\Rightarrow n(A\times B)=4$

It is known that if C is a set with $n(C)=m$, then $n[P(C)]=2^m$

Therefore, the set $A\times B$ has $2^4=16$ subsets.

These are,

$\Phi,\{(1,3)\},\{(1,4)\},\{(2,3)\},\{(2,4)\},\{(1,3),(1,4)\},\{(1,3),(2,3)\},\{(1,3),(2,4)\},,$
$\{(1,4),(2,3)\},\{(1,4),(2,4)\},\{(2,3),(2,4)\},\{(1,3),(1,4),(2,3)\},\{(1,3),(1,4),(2,4)\},,$
$\{(1,3),(2,3),(2,4)\},\{(1,4),(2,3),(2,4)\},\{(1,3),(1,4),(2,3),(2,4)\}$

Question 9. Let A and B be two sets such that $n(A)=3$ and $n(B)=2$. If $(x,1),(y,2)$, $(z,1)$ are in $A\times B$, find A and B, where x, y and z are distinct elements.

Solution : Given that, $n(A)=3$ and $n(B)=2$.

It is also given that, $(x,1),(y,2),(z,1)$ are in $A\times B$.

It is known that, A is the set of first elements of the ordered pair elements of $A\times B$.

B is the set of second elements of the ordered pair elements of $A\times B$.

Thus, x, y, and z are the elements of A and $1,2$ are the elements of B.

Since $n(A)=3$ and $n(B)=2$, it is clear that $A=\{x,y,z\}$ and $B=\{1,2\}$.

Question 10. The Cartesian product $A\times A$ has 9 elements among which are found $(-1,0)$ and $(0,1)$. Find the set A and the remaining elements of $A\times A$.

Solution : It is known that if $n(A)=p$ and $n(B)=q$, then $n(A\times B)=pq$

$n(A\times A)=n(A)\times n(A)$

It is given that $n(A\times A)=9$.

Thus, $n(A)\times n(A)=9$

$\Rightarrow n(A)=3$

Thus, the ordered pairs $(-1,0)$ and $(0,1)$ are two of the nine elements of $A\times A$.

It is also known that $A \times A = \{(a,a) : a \in A\}$.

So, $-1, 0$, and 1 are elements of A.

Since $n(A)=3$, it is clear that $A=\{-1,0,1\}$.

Therefore, the remaining elements of set $A \times A$ are $(-1,-1),(-1,1),(0,-1),(0,0)$, $(1,-1),(1,0)$, and $(1,1)$.

Exercise 2.2

Question 1. Let $A = \{1,2,3\ldots14\}$. Define a relation R from A to A by $R = \{(x, y) : 3x - y = 0$, where $x, y \in A\}$. Write down its domain, codomain and range.

Solution : The relation R from A to A is given as $R = \{(x, y) : 3x - y = 0$, where $x, y \in A\}$

That is, $R = \{(x, y) : 3x = y$, where $x, y \in A\}$

Thus, $R = \{(1,3),(2,6),(3,9),(4,12)\}$

The domain of R is the set of all first elements of the ordered pairs in the relation.

Therefore, Domain of $R = \{1,2,3,4\}$

The complete set A is the codomain of the relation R.

Therefore, Codomain of $R = A = \{1,2,3\ldots14\}$

The range of R is the set of all second elements of the ordered pairs in the relation.

Therefore, Range of $R = \{3,6,9,12\}$

Question 2. Define a relation R on the set $\mathbf{N}$ of natural numbers by $R = \{(x, y) : y = x + 5, x$ is a natural number less than $4; x, y \in \mathbf{N}\}$. Depict this relationship using roster form. Write down the domain and the range.

Solution : $R = \{(x, y) : y = x + 5, x$ is a natural number less than $4, x, y \in \mathbf{N}\}$ The natural numbers less than 4 are $1, 2$, and 3.

Thus, the relation is $R = \{(1,6),(2,7),(3,8)\}$.

The domain of R is the set of all first elements of the ordered pairs in the relation.

Therefore, Domain of $R = \{1,2,3\}$

The range of R is the set of all second elements of the ordered pairs in the relation. Therefore, Range of $R = \{6,7,8\}$

Question 3. $A=\{1,2,3,5\}$ and $B=\{4,6,9\}$. Define a relation R from A to B by

$R=\{(x,y):$ the difference between x and y is odd; $x\in A, y\in B\}$. Write R in roster form.

Solution : Given that, $A=\{1,2,3,5\}$ and $B=\{4,6,9\}$.

$R=\{(x,y):$ the difference between x and y is odd; $x\in A, y\in B\}$

Therefore, $R=\{(1,4),(1,6),(2,9),(3,4),(3,6),(5,4),(5,6)\}$

Question 4. The given figure shows a relationship between the sets P and Q. Write this relation (i) in set-builder form (ii) in roster form. What is its domain and range?

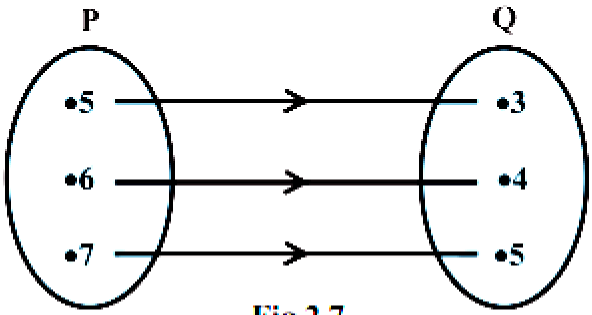


Fig 2.7

Solution : From the figure, $P=\{5,6,7\}$ and $Q=\{3,4,5\}$.

(i) The relation in set-builder form is,

$R=\{(x,y): y=x-2; x\in P\}$ or $R=\{(x,y): y=x-2$ for $x=5,6,7\}$

(ii) The relation in roster form is,

$R=\{(5,3),(6,4),(7,5)\}$

Domain of $R=\{5,6,7\}$

Range of $R=\{3,4,5\}$

Question 5. Let $A=\{1,2,3,4,6\}$. Let R be the relation on A defined by $\{(a,b): a,b \in A,\ b$ is exactly divisible by $a\}$.

(i) Write R in roster form

(ii) Find the domain of R

(iii) Find the range of R.

Solution : Given that, $A=\{1,2,3,4,6\}$ and $R=\{(a,b): a,b\in A, b$ is exactly divisible by $a\}$.

(i) R in roster form is,

$R=\{(1,1),(1,2),(1,3),(1,4),(1,6),(2,2),(2,4),(2,6),(3,3)(3,6),(4,4),(6,6)\}$,

(ii) The domain of R is the set of all first elements of the ordered pairs in the relation.

Therefore, Domain of $R=\{1,2,3,4,6\}$

(iii) The range of R is the set of all second elements of the ordered pairs in the relation.

Therefore, Range of $R=\{1,2,3,4,6\}$

Question 6. Determine the domain and range of the relation R defined by $R=\{(x,x+5):x\in\{0,1,2,3,4,5\}\}$.

Solution : Given that, $R=\{(x,x+5):x\in\{0,1,2,3,4,5\}\}$

Thus, $R=\{(0,5),(1,6),(2,7),(3,8),(4,9),(5,10)\}$

The domain of R is the set of all first elements of the ordered pairs in the relation. Therefore, Domain of $R=\{0,1,2,3,4,5\}$

The range of R is the set of all second elements of the ordered pairs in the relation.

Therefore, Range of $R=\{5,6,7,8,9,10\}$

Question 7. Write the relation $R=\{(x,x^3):x$ is a prime number less than $10\}$ in roster form.

Solution : The relation is $R=\{(x,x^3):x$ is a prime number less than $10\}$.

The prime numbers less than 10 are $2,3,5,$ and 7.

Therefore, the relation in roster form is $R=\{(2,8),(3,27),(5,125),(7,343)\}$.

Question 8. Let $A=\{x,y,z\}$ and $B=\{1,2\}$. Find the number of relations from A to B.

Solution : Given that $A=\{x,y,z\}$ and $B=\{1,2\}$.

Now, $A\times B=\{(x,1),(x,2),(y,1),(y,2),(z,1),(z,2)\}$

Here, $n(A\times B)=6$.

So, the number of subsets of $A\times B$ is 2^6.

Therefore, the number of relations from A to is 2^6.

Question 9. Let R be the relation on $\mathbf{Z}$ defined by $R=\{(a,b):a,b\in\mathbf{Z}, a-b$ is an integer $\}$. Find the domain and range of R.

Solution : Given that, $R = \{(a,b): a,b \in \mathbf{Z}, a-b$ is an integer $\}$

It is known that the difference between any two integers will be an integer.

Therefore, Domain of $R = \mathbf{Z}$ and Range of $R = \mathbf{Z}$.

Exercise 2.3

Question 1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) $\{(2,1),(5,1),(8,1),(11,1),(14,1),(17,1)\}$

(ii) $\{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(14,7)\}$

(iii) $\{(1,3),(1,5),(2,5)\}$

Solution :

(i) The relation is $\{(2,1),(5,1),(8,1),(11,1),(14,1),(17,1)\}$.

This relation is a function. Because, $2,5,8,11,14,$ and 17 are the elements of the domain of the given relation having their unique images.

Here, Domain $=\{2,5,8,11,14,17\}$ and Range $=\{1\}$

(ii) $\{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(14,7)\}$

This relation is a function. Because, $2,4,6,8,10,12,$ and 14 are the elements of the domain of the given relation having their unique images.

Here, Domain $=\{2,4,6,8,10,12,14\}$ and Range $=\{1,2,3,4,5,6,7\}$

(iii) $\{(1,3),(1,5),(2,5)\}$

This relation is not a function. Because, the same first element that is, 1 corresponds to two different images 3 and 5.

Question 2. Find the domain and range of the following real function:

(i) $f(x) = -|x|$

(ii) $f(x) = \sqrt{9-x^2}$

Solution :

(i) $f(x) = -|x|, x \in R$

It is known that, $|x| = \begin{Bmatrix} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{Bmatrix}$

Thus, $f(x) = -|x| = \begin{cases} -x, \text{if } x \geq 0 \\ x, \text{if } x < 0 \end{cases}$

Since $f(x)$ is defined for $x \in \mathbf{R}$, the domain of f is $\mathbf{R}$.

It can be observed that the range of $f(x) = -|x|$ is all real numbers except positive real numbers.

Therefore, the range of f is $(-\infty, 0]$.

(ii) $f(x) = \sqrt{9 - x^2}$

Since $\sqrt{9-x^2}$ is defined for all real numbers that are greater than or equal to -3 and less than or equal to 3, the domain of $f(x)$ is $\{x : -3 \leq x \leq 3\}$ or $[-3, 3]$.

For any value of x such that $-3 \leq x \leq 3$, the value of $f(x)$ will lie between 0 and 3. The range of $f(x)$ is $\{x : 0 \leq x \leq 3\}$ or $[0, 3]$.

Question 3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

(i) $f(0)$,

(ii) $f(7)$,

(iii) $f(-3)$

Solution : The given function is $f(x) = 2x - 5$.

(i) Substitute $x = 0$,

$f(0) = 2 \times 0 - 5 = 0 - 5 = -5$

(ii) Substitute $x = 7$,

$f(7) = 2 \times 7 - 5 = 14 - 5 = 9$

(iii) Substitute $x = -3$,

$f(-3) = 2 \times (-3) - 5 = -6 - 5 = -11$

Question 4. The function $'t'$ which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(\mathrm{C}) = \frac{9\mathrm{C}}{5} + 32$. Find

(i) $t(0)$

(ii) $t(28)$

(iii) $t(-10)$

(iv) The value of C, when $t(\mathrm{C})=212$

Solution : The given function is $t(\mathrm{C})=\frac{9\mathrm{C}}{5}+32$.

(i) Substitute $\mathrm{C}=0$,

$$t(0)=\frac{9\times 0}{5}+32=0+32=32$$

(ii) Substitute $\mathrm{C}=28$,

$$t(28)=\frac{9\times 28}{5}+32=\frac{252+160}{5}=\frac{412}{5}$$

(iii) Substitute $\mathrm{C}=-10$,

$$t(-10)=\frac{9\times(-10)}{5}+32=9\times(-2)+32=-18+32=14$$

(iv) It is given that $t(\mathrm{C})=212$

$$212=\frac{9\mathrm{C}}{5}+32$$

Subtract 32 from both sides,

$$\Rightarrow \frac{9\mathrm{C}}{5}=212-32$$

Subtract the numbers,

$$\Rightarrow \frac{9\mathrm{C}}{5}=180$$

Multiply both sides by 5,

$$\Rightarrow 9\mathrm{C}=180\times 5$$

$$\Rightarrow C=\frac{180\times 5}{9}=100$$

Therefore, the value of t, when $t(\mathrm{C})=212$ is 100.

Question 5. Find the range of each of the following functions.

(i) $f(x)=2-3x, x\in\mathbf{R}, x>0$.

(ii) $f(x)=x^2+2, x$, is a real number.

(iii) $f(x)=x, x$ is a real number

Solution :

(i) $f(x) = 2 - 3x, x \in \mathbf{R}, x > 0$

The values of $f(x)$ for various values of real numbers $x > 0$ can be written in the tabular form as,

x	0.01	0.1	0.9	1	2	2.5	4	5	…..
$f(x)$	1.97	1.7	–0.7	–1	–4	–5.5	–10	–13	…..

Therefore, it can be clearly observed that the range of f is the set of all real numbers less than 2.

Therefore, range of $f = (-\infty, 2)$.

(ii) $f(x) = x^2 + 2, x$, is a real number

The values of $f(x)$ for various values of real numbers x can be written in the tabular form as,

x	0	±0.3	±0.8	±1	±2	±3	…..
$f(x)$	2	2.09	2.64	3	6	11	…..

Therefore, it can be clearly observed that the range of f is the set of all real numbers greater than 2.

Therefore, range of $f = (2, \infty)$.

(iii) $f(x) = x, x$ is a real number

It is clear that the range of f is the set of all real numbers.

Therefore, Range of $f = \mathbf{R}$.

Miscellaneous Exercises

Question 1. The relation f is defined by $f(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10 \end{cases}$. Show that f is a function and g is not a function.

Solution : Here, the relation f is defined as $f(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10 \end{cases}$.

It is observed that for,

$$\begin{aligned} &0 \le x < 3, \quad f(x) = x^2 \\ &3 < x \le 10, \quad f(x) = 3x \end{aligned}$$

Also, at $x = 3$, $f(x) = 3^2 = 9$

Or $f(x) = 3 \times 3 = 9$ that is, at $x = 3, f(x) = 9$.

Thus, for $0 \leq x \leq 10$, the images of $f(x)$ are unique.

Therefore, the given relation is a function.

The relation g is defined as $g(x) = \begin{cases} x^2, 0 \leq x \leq 2 \\ 3x, 2 \leq x \leq 10 \end{cases}$.

It can be observed that for, $x = 2$, $g(x) = 2^2 = 4$ and $g(x) = 3 \times 2 = 6$.

Thus, element 2 of the domain of the relation g corresponds to two different images that is, 4 and 6.

Therefore, this relation is not a function.

Question 2. If $f(x) = x^2$, find. $\frac{f(1.1) - f(1)}{(1.1 - 1)}$

Solution : Given that, $f(x) = x^2$.

Now, $\frac{f(1.1) - f(1)}{(1.1 - 1)} = \frac{(1.1)^2 - (1)^2}{(1.1 - 1)}$

Simplify the squares,

$$= \frac{1.21 - 1}{0.1}$$

Subtract the numbers,

$$= \frac{0.21}{0.1}$$

Divide the numbers,

$$= 2.1$$

Therefore, $\frac{f(1.1) - f(1)}{(1.1 - 1)} = 2.1$.

Question 3. Find the domain of the function $f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

Solution : Here, the function is $f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

Simplify $x^2-8x+12$,

$$f(x)=\frac{x^2+2x+1}{x^2-8x+12}=\frac{x^2+2x+1}{(x-6)(x-2)}$$

It can be observed that function f is defined for all real numbers except at $x=6$ and $x=2$.

Therefore, the domain of f is $\mathbf{R}-\{2,6\}$.

Question 4. Find the domain and the range of the real function f defined by $f(x)=\sqrt{(x-1)}$

Solution : Here, the real function is $f(x)=\sqrt{(x-1)}$.

It can be observed that $\sqrt{(x-1)}$ is defined for $x\geq 1$.

Thus, the domain of f is the set of all real numbers greater than 0 equal to 1.

That is, the domain of $f=[1,\infty)$.

As $x\geq 1\Rightarrow (x-1)\geq 0\Rightarrow \sqrt{(x-1)}\geq 0$

Therefore, the range of f is the set of all real numbers greater than or equal to 0. Therefore, the range of $f=[0,\infty)$.

Question 5. Find the domain and the range of the real function f defined by $f(x)=|x-1|$.

Solution : Here, the real function is $f(x)=|x-1|$.

It is clear that $|x-1|$ is defined for all real numbers.

Thus, Domain of $f=\mathbf{R}$.

And for $x\in\mathbf{R}, |x-1|$ assumes all real numbers.

Therefore, the range of f is the set of all non-negative real numbers.

Question 6. Let $f=\left\{\left(x,\frac{x^2}{1+x^2}\right):x\in\mathbf{R}\right\}$ be a function from $\mathbf{R}$ into $\mathbf{R}$. Determine the range of f.

Solution : Here,

$$f=\left\{\left(x,\frac{x^2}{1+x^2}\right):x\in\mathbf{R}\right\}=\left\{(0,0),\left(\pm 0.5,\frac{1}{5}\right),\left(\pm 1,\frac{1}{2}\right),\left(\pm 1.5,\frac{9}{13}\right),\left(\pm 2,\frac{4}{5}\right),\left(3,\frac{9}{10}\right),\left(4,\frac{16}{17}\right),\ldots\right\}$$

The range of f is the set of all second elements.

It can be seen that all these elements are greater than or equal to 0 but less than 1. Therefore, range of $f = [0,1)$.

Question 7. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined, respectively by $f(x) = x+1, g(x) = 2x-3$. Find $f+g, f-g$ and $\frac{f}{g}$.

Solution : Here, $f, g : \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = x+1, g(x) = 2x-3$.

$(f+g)(x) = f(x)+g(x) = (x+1)+(2x-3) = 3x-2$

Thus, $(f+g)(x) = 3x-2$

$(f-g)(x) = f(x)-g(x) = (x+1)-(2x-3) = x+1-2x+3 = -x+4$

Thus, $(f-g)(x) = -x+4$

Now, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0, x \in \mathbf{R}$

$\left(\frac{f}{g}\right)(x) = \frac{x+1}{2x-3}, 2x-3 \neq 0$ or $2x \neq 3$

Therefore, $\left(\frac{f}{g}\right)(x) = \frac{x+1}{2x-3}, x \neq \frac{3}{2}$

Question 8. Let $f = \{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a function from $\mathbf{Z}$ to $\mathbf{Z}$ defined by $f(x) = ax+b$, for some integers a, b. Determine a, b.

Solution : Given that, $f = \{(1,1),(2,3),(0,-1),(-1,-3)\}$ and $f(x) = ax+b$.

Consider the pairs $(1,1)$ and $(0,-1)$,

$(1,1) \in f \Rightarrow f(1) = 1 \Rightarrow a \times 1 + b = 1 \Rightarrow a+b = 1$

$(0,-1) \in f \Rightarrow f(0) = -1 \Rightarrow a \times 0 + b = -1 \Rightarrow b = -1$

Substitute $b = -1$ in $a+b = 1$,

That is, $a + (-1) = 1 \Rightarrow a = 1+1 = 2$.

Therefore, the respective values of a and b are 2 and -1.

Question 9. Let R be a relation from $\mathbf{N}$ to $\mathbf{N}$ defined by $R = \{(a,b) : a, b \in \mathbf{N}$ and $a = b^2\}$. Are the following true?

(i) $(a,a) \in R$, for all $a \in \mathbf{N}$

(ii) $(a,b) \in R$, implies $(b,a) \in R$

(iii) $(a,b) \in R, (b,c) \in R$ implies $(a,c) \in R$.

Justify your answer in each case.

Solution : Given that, $R = \{(a,b) : a,b \in \mathbf{N} \text{ and } a = b^2\}$.

(i) It can be observed that $2 \in \mathbf{N}$; but $2 \neq 2^2 = 4$.

Thus, the statement "$(a,a) \in R$, for all $a \in \mathbf{N}$" is not true.

(ii) It can be observed that $(9,3) \in \mathbf{N}$ because $9,3 \in \mathbf{N}$ and $9 = 3^2$.

Then, $3 \neq 9^2 = 81$

Thus, $(3,9) \notin \mathbf{N}$

Therefore, the statement " $(a,b) \in R$, implies $(b,a) \in R''$ is not true.

(iii) It can be observed that $(9,3) \in R, (16,4) \in R$ because $9,3,16,4 \in \mathbf{N}$ and $9 = 3^2$ also $16 = 4^2$.

Now, $9 \neq 4^2 = 16$; therefore, $(9,4) \notin \mathbf{N}$

Therefore, the statement " $(a,b) \in R, (b,c) \in R$ implies $(a,c) \in R''$ is not true.

Question 10. Let $A=\{1,2,3,4\}, B=\{1,5,9,11,15,16\}$ and $f = \{(1,5),(2,9),(3,1),(4,5),(2,11)\}$ Are the following true?

(i) f is a relation from A to B.

(ii) f is a function from A to B.

Justify your answer in each case.

Solution : Here, $A=\{1,2,3,4\}$ and $B=\{1,5,9,11,15,16\}$.

Now,

$$A \times B = \{(1,1),(1,5),(1,9),(1,11),(1,15),(1,16),(2,1),(2,5),(2,9),(2,11),\\ (2,15),(2,16),(3,1),(3,5),(3,9),(3,11),(3,15),(3,16),(4,1),(4,5),(4,9),,\\ (4,11),(4,15),(4,16)\}$$

It is given that $f = \{(1,5),(2,9),(3,1),(4,5),(2,11)\}$.

(i) A relation from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A \times B$.

It is seen that f is a subset of $A \times B$.

Therefore, f is a relation from A to B.

(ii) Because the same first element that is, 2 corresponds to two different images 9 and 11, relation f is not a function.

Question 11. Let f be the subset of $\mathbf{Z}\times\mathbf{Z}$ defined by $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? justify your answer.

Solution : Here, the relation f is defined as $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$.

It is known that a relation f from a set A to B is said to be a function if every element of set A has unique images in set B.

Since $2,6,-2,-6\in\mathbf{Z},(2\times6,2+6),(-2\times-6,-2+\ (-6))\in f$

That is, $(12,8),(12,-8)\in f$

It can be observed that the same first element 12 corresponds to two different images 8 and -8.

Therefore, relation f is not a function.

Question 12. Let $\mathrm{A}=\{9,10,11,12,13\}$ and let $f:\mathrm{A}\to\mathbf{N}$ be defined by $f(n)=$ the highest prime factor of n. Find the range of f.

Solution : Here, $\mathrm{A}=\{9,10,11,12,13\}$ and $f:\mathrm{A}\to\mathbf{N}$ is defined as $f(n)=$ The highest prime factor of n.

Determine the prime factor of each number,

Prime factor of $9=3$

Prime factors of $10=2,5$

Prime factor of $11=11$

Prime factors of $12=2,3$

Prime factor of $13=13$

Determine the highest prime factor of each number,

$f(9)=$ The highest prime factor of $9=3$

$f(10)=$ The highest prime factor of $10=5$

$f(11)=$ The highest prime factor of $11=11$

$f(12)=$ The highest prime factor of $12=3$

$f(13)=$ The highest prime factor of $13=13$

The range of f is the set of all $f(n)$, where $n \in A$.

Therefore, Range of $f = \{3, 5, 11, 13\}$.