Example 1

Convert $40^{\circ}20'$ into radian measure.

Solution

Given that $40^{\circ}20' = 40\frac{1}{3}$ degree

We know that $180^{\circ} = \pi$ radian.

$= \frac{\pi}{180} \times \frac{121}{3}$ radian

$= \frac{121\pi}{540}$ radian.

Therefore $40^{\circ}20' = \frac{121\pi}{540}$ radian.

Example 2

Convert 6 radians into degree measure.

Solution

Given that π radian $= 180^{\circ}$.

Hence

6 radians $= \frac{180}{\pi} \times 6$ degree

$= \frac{1080 \times 7}{22}$ degree

Take $1^{\circ} = 60'$

$= 343\frac{7}{11}$ degree $= 343^{\circ} + \frac{7 \times 60}{11}$ minute

$= 343^{\circ} + 38' + \frac{2}{11}$ minute

$= 343^{\circ}38'11''$ The answer

Example 3

Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4cm (use $\pi=\frac{22}{7}$).

Solution

Given that

$$l=37.4\text{cm} \text{ and } \theta=60^{\circ}=\frac{60\pi}{180}$$

$$\text{radian }=\frac{\pi}{3}$$

The value $r=\frac{l}{\theta}$,

Solving we get

$$r=\frac{37.4\times3}{\pi}=\frac{37.4\times3\times7}{22}=35.7\text{cm}$$

Example 4

The minute hand of a watch is 1.5cm long. How far does its tip move in 40 minutes? (Use $\pi=3.14$).

Solution

Given that

Watch is 1.5cm long

It complete 60 revolution in one minute.

in 40 minute,

minute hand will turn $\frac{2}{3}$ of a revolution

ie, $\theta=\frac{2}{3}\times360^{\circ}$

or $\frac{4\pi}{3}$ radian.

The distance $l=r\theta=1.5\times\frac{4\pi}{3}\text{cm}=2\pi\text{cm}=2\times3.14\text{cm}=6.28\text{cm}$

Example 5

If the arcs of the same lengths in two circles subtend angles 65° and 110° at the centre, find the ratio of their radii.

Solution

Given that 65° and 110°

Let r_1 and r_2 be the radii of the two circles.

Given that $\theta_1 = 65^{\circ} = \frac{\pi}{180} \times 65 = \frac{13\pi}{36}$ radian and $\theta_2 = 110^{\circ} = \frac{\pi}{180} \times 110 = \frac{22\pi}{36}$ radian

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2$, i.e., $\frac{r_1}{r_2} = \frac{22}{13}$

Hence $r_1 : r_2 = 22:13$.

Exercise 3.1

Question 1:

Find the radian measures corresponding to the following degree measures:

(i) 25°

(ii) $-47^{\circ}30'$

(iii) 240°

(iv) 520°

Answer 1:

(i) 25°

We know that $180^{\circ} = \pi$ radian

$\therefore 25^{\circ} = \frac{\pi}{180} \times 25$ radian $= \frac{5\pi}{36}$ radian

(ii) $-47^{\circ}30'$

$-47^{\circ}30' - 47\frac{1}{2} = \frac{-95}{2}$ deqree

Since 180° = π radian

$\frac{-95}{2}$ deg ree $=\frac{\pi}{180}\times\left(\frac{-95}{2}\right)$ radian $=\left(\frac{-19}{36\times 2}\right)\pi$ radian $=\frac{-19}{72}\pi$ radian

$\therefore -47^{\circ}30' = \frac{-19}{72}\pi$ radian

(iii) 240°

We know that $180^{\circ} = \pi$ radian

$\therefore 240^{\circ} = \frac{\pi}{180}\times 240$ radian $= \frac{4}{3}\pi$ radian

(iv) 520°

We know that $180^{\circ} = \pi$ radian

$\therefore 520^{\circ} = \frac{\pi}{180}\times 520$ radian $= \frac{26\pi}{9}$ radian

Question 2:

Find the degree measures corresponding to the following radian measures $\left(\text{Use } \pi = \frac{22}{7}\right)$

(i) $\frac{11}{16}$

(ii) -4

(iii) $\frac{5\pi}{3}$

(iv) $\frac{7\pi}{6}$

Answer 2:

(i) $\frac{11}{16}$

We know that n radian $=180^{\circ}$

$\therefore \frac{11}{16}$ radian $= \frac{180}{\pi}\times\frac{11}{16}$ degree $= \frac{45\times 11}{\pi\times 4}$ degree

$= \frac{45\times 11\times 7}{22\times 4}$ degree

$=\frac{315}{8}$ degree

$=39\frac{3}{8}$ degree

$=39^{\circ}+\frac{3\times 60}{8}$ minutes

$=39^{\circ}+22'+\frac{1}{2}$ minutes

$=39^{\circ}22'30^{\circ}$

$\therefore \frac{11}{16}$ radain $=\frac{180}{\pi}\times\frac{11}{16}$ degree $=\frac{45\times 11}{\pi\times 4}$ degree

$=\frac{45\times 11\times 7}{22\times 4}$ degree $=\frac{315}{8}$ degree

$=39\frac{3}{8}$ degree

$=39^{\circ}+\frac{3\times 60}{8}$ minutes $\left[1^{\circ}=60'\right]$

$=39^{\circ}+22'+\frac{1}{2}$ minutes

$=39^{\circ}22'30'$

(ii) -4

We know that n radian $=180^{\circ}$

-4 radian $=\frac{180}{\pi}\times(-4)$ degree $=\frac{180\times 7(-4)}{22}$ degree

$=\frac{-2520}{11}$ degree $=-229\frac{1}{11}$ degree

$=-229^{\circ}+\frac{1\times 60}{11}$ minutes $\left[1^{\circ}=60'\right]$

$=-229^{\circ}+5'+\frac{5}{11}$ minutes

$=-229^{\circ}5'27'$ $\left[1'=60''\right]$

(iii) $\frac{5\pi}{3}$

We know that π radian $=180^{\circ}$

$\therefore \frac{5\pi}{3}$ radian $=\frac{180}{\pi}\times\frac{5\pi}{3}$ degree $=300^{\circ}$

(iv) $\frac{7\pi}{6}$

We know that π radian $=180^{\circ}$

$\therefore \frac{7\pi}{6}$ radian $=\frac{180}{\pi}\times\frac{7\pi}{6}=210^{\circ}$

Question 3:

A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Answer 3:

Given that

Revolution in 1 minute $=360$

$\therefore$ in 1 second $=\frac{360}{60}=6$

In one complete revolution,

the wheel turns an angle of 2π radian.

in 6 complete revolutions, it will turn an angle of $6\times2\pi$ radian,

i.e., 12 π radian

Thus, in one second, the wheel turns an angle of 12π radian.

Question 4:

Find the degree measure of the angle subtended at the centre of a circle of radius $100\,\text{cm}$ by an arc of length $22\,\text{cm}$.

$\left(\text{Use } \pi=\frac{22}{7}\right)$

Answer 4:

We know that in a circle of radius r unit, if an arc of length / unit subtends an angle θ radian at the centre, then

$\theta = \frac{1}{r}$

Therefore, forr $= 100\,\text{cm}, \text{I} = 22\,\text{cm}$, we have

$$\theta = \frac{22}{100} \text{ radian } = \frac{180}{\pi} \times \frac{22}{100} \text{ degree } = \frac{180 \times 7 \times 22}{22 \times 100} \text{ degree}$$

$$= \frac{126}{10} \text{ degree } = 12\frac{3}{5} \text{ degree } = 12^{\circ}36' \quad \left[1^{\circ} = 60'\right]$$

Thus, the required angle is $12^{\circ}36'$.

Question 5:

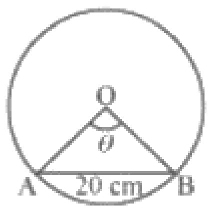
In a circle of diameter $40\,\text{cm}$, the length of a chord is $20\,\text{cm}$. Find the length of minor arc of the chord.

Answer 5:

Diameter of the circle $= 40\,\text{cm}$

$\therefore$ Radius (r) of the circle $= \frac{40}{2}\text{cm} = 20\,\text{cm}$

Let A B be a chord (length $= 20\,\text{cm}$) of the circle.



In $\triangle \text{OAB}$, $\text{OA} = \text{OB} =$ Radius of circle $= 20\,\text{cm}$

Also, $AB = 20\,\text{cm}$

Thus, $\triangle \text{OAB}$ is an equilateral triangle.

$\therefore \theta = 60^{\circ} = \frac{\pi}{3}$ radian

We know that in a circle of radius r unit,

if an arc of length unit subtends an angle θ

$\theta = \frac{l}{r}$

$$\frac{\pi}{3} = \frac{\text{AB}}{20} \Rightarrow \text{AB} = \frac{20\pi}{3}\text{cm}$$

Thus, the length of the minor arc of the chord is $\frac{20\pi}{3}$ cm

Question 6:

If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Answer 6:

Given that

Let the radii of the two circles be r_1 and r_2.

Let an arc of length subtend an angle of 60° at the centre of the circle of radius r_1, while let an arc of length subtend an angle of 75° at the centre of the circle of radius r_2.

Now, $60^{\circ} = \frac{\pi}{3}$ radian and $75^{\circ} = \frac{5\pi}{12}$ radian

We know that in a circle of radius r unit, if an arc of length unit subtends an angle θ

$$\theta = \frac{l}{r} \text{ or } l = r\theta$$

$$\therefore l = \frac{r_1\pi}{3} \text{ and } l = \frac{r_2 5\pi}{12}$$

$$\Rightarrow \frac{r_1\pi}{3} = \frac{r_2 5\pi}{12}$$

$$\Rightarrow r_1 = \frac{r_2 5}{4}$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{5}{4}$$

Thus, the ratio of the radii is 5: 4

Question 7:

Find the angle in radian though which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm

(ii) 15 cm

(iii) 21 cm

Answer 7:

Given that

in a circle of radius r unit, if an arc of length unit subtends an angle θ radian at the centre, then $\theta = \frac{l}{r}$ It is given that $r = 75\text{cm}$

(i) Here, $I = 10\text{cm}$

$$\theta = \frac{10}{75} \text{ radian } = \frac{2}{15} \text{ radian}$$

(ii) Here, $I = 15\text{cm}$

$$\theta = \frac{15}{75} \text{radian} = \frac{1}{5} \text{ radian}$$

(iii) Here, $I = 21\text{cm}$

$$\theta = \frac{21}{75} \text{ radian } = \frac{7}{25} \text{ radian}$$

Example 6

If $\cos x = -\frac{3}{5}, x$ lies in the third quadrant, find the values of other five trigonometric functions.

Solution

Given that

$$\cos x = -\frac{3}{5},$$

we have $\sec x = -\frac{5}{3}$

Now $\quad \sin^2 x + \cos^2 x = 1,$

i.e., $\sin^2 x = 1 - \cos^2 x$

or $\quad \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\quad \sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative.

Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}$$

Example 7

If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution

Given that

$$\cot x = -\frac{5}{12},$$

we have $\tan x = -\frac{12}{5}$

Now

$$\sec^2 x = 1 + \tan^2 x$$

$$= 1 + \frac{144}{25}$$

$$= \frac{169}{25}$$

Hence

$$\sec x = \pm\frac{13}{5}$$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5}$$

It gives

$$\cos x = -\frac{5}{13}$$

we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $\quad \operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}$

Example 8

Find the value of $\sin\frac{31\pi}{3}$.

Solution

Given that

$\sin x$ repeats after an interval of 2π.

Therefore

$$\sin\frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right)$$

$$= \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}$$

Example 9

Find the value of $\cos\left(-1710^{\circ}\right)$,

Solution

We know that

$\cos x$ repeats after an interval of 2π or 360°.

Therefore,

Rewrite the function as

$$\cos\left(-1710^{\circ}\right) = \cos\left(-1710^{\circ} + 5 \times 360^{\circ}\right)$$

$$= \cos\left(-1710^{\circ} + 1800^{\circ}\right)$$

$$= \cos 90^{\circ} = 0.$$

Exercise 3.2

Question 1:

Find the values of other five trigonometric functions if $\cos x = -\frac{1}{2}, x$ lies in third quadrant.

Answer 1:

Given that

$$\cos x = -\frac{1}{2}$$

$$\therefore \sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{1}{2}\right)} = -2$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^2 x = 1 - \left(-\frac{1}{2}\right)^2$$

$$\Rightarrow \sin^2 x = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$$

Since x lies in the 3^{rd} quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x = -\frac{\sqrt{3}}{2}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)} = -\frac{2}{\sqrt{3}}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)} = \sqrt{3}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sqrt{3}}$$

Question 2:

Find the values of other five trigonometric functions if $\sin x = \frac{3}{5}$, x lies in second quadrant.

Answer 2:

Given that

$$\sin x = \frac{3}{5}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \cos^2 x = 1 - \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \cos^2 x = 1 - \frac{9}{25}$$

$$\Rightarrow \cos^2 x = \frac{16}{25}$$

$$\Rightarrow \cos x = \pm\frac{4}{5}$$

Since x lies in the 2^{nd} quadrant, the value of $\cos x$ will be negative

$$\therefore \cos x = -\frac{4}{5}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{4}{5}\right)} = -\frac{5}{4}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)} = -\frac{3}{4}$$

$$\cot x = \frac{1}{\tan x} = -\frac{4}{3}$$

Question 3:

Find the values of other five trigonometric functions if $\cot x = \frac{3}{4}, x$ lies in third quadrant.

Answer 3:

Given that

$$\cot x = \frac{3}{4}$$

$$\tan x = \frac{1}{\cot x} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(\frac{4}{3}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{16}{9} = \sec^2 x$$

$$\Rightarrow \frac{25}{9} = \sec^2 x$$

$$\Rightarrow \sec x = \pm\frac{5}{3}$$

Since x lies in the 3^{rd} quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{5}{3}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(-\frac{5}{3}\right)} = -\frac{3}{5}$$

$$\tan x=\frac{\sin x}{\cos x}$$

$$\Rightarrow \frac{4}{3}=\frac{\sin x}{\left(\frac{-3}{5}\right)}$$

$$\Rightarrow \sin x=\left(\frac{4}{3}\right)\times\left(\frac{-3}{5}\right)=-\frac{4}{5}$$

$$\operatorname{cosec} x=\frac{1}{\sin x}=-\frac{5}{4}$$

Question 4:

Find the values of other five trigonometric functions if $\sec x=\frac{13}{5}, x$ lies in fourth quadrant.

Answer 4:

Given that

$$\sec x=\frac{13}{5}$$

$$\cos x=\frac{1}{\sec x}=\frac{1}{\left(\frac{13}{5}\right)}=\frac{5}{13}$$

$$\sin^2 x+\cos^2 x=1$$

$$\Rightarrow \sin^2 x=1-\cos^2 x$$

$$\Rightarrow \sin^2 x=1-\left(\frac{5}{13}\right)^2$$

$$\Rightarrow \sin^2 x=1-\frac{25}{169}=\frac{144}{169}$$

$$\Rightarrow \sin x=\pm\frac{12}{13}$$

Since x lies in the 4^{th} quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x=-\frac{12}{13}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{12}{13}\right)} = -\frac{13}{12}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(\frac{-12}{13}\right)}{\left(\frac{5}{13}\right)} = -\frac{12}{5}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{12}{5}\right)} = -\frac{5}{12}$$

Question 5:

Find the values of other five trigonometric functions if $\tan x = -\frac{5}{12}, x$ lies in second quadrant.

Answer 5:

Given that

$$\tan x = -\frac{5}{12}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{5}{12}\right)} = -\frac{12}{5}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(-\frac{5}{12}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{25}{144} = \sec^2 x$$

$$\Rightarrow \frac{169}{144} = \sec^2 x$$

$$\Rightarrow \sec x = \pm\frac{13}{12}$$

Since x lies in the 2^{nd} quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{13}{12}$$

$$\cos x=\frac{1}{\sec x}=\frac{1}{\left(-\frac{13}{12}\right)}=-\frac{12}{13}$$

$$\tan x=\frac{\sin x}{\cos x}$$

$$\Rightarrow -\frac{5}{12}=\frac{\sin x}{\left(-\frac{12}{13}\right)}$$

$$\Rightarrow \sin x=\left(-\frac{5}{12}\right)\times\left(-\frac{12}{13}\right)=\frac{5}{13}$$

$$\operatorname{cosec} x=\frac{1}{\sin x}=\frac{1}{\left(\frac{5}{13}\right)}=\frac{13}{5}$$

Question 6:

Find the value of the trigonometric function $\sin 765^{\circ}$

Answer 6:

It is known that the values of $\sin x$ repeat after an interval of 2π or 360°.

$$\therefore \sin 765^{\circ}=\sin\left(2\times 360^{\circ}+45^{\circ}\right)$$

$$=\sin 45^{\circ}=\frac{1}{\sqrt{2}}$$

Question 7:

Find the value of the trigonometric function $\operatorname{cosec}\left(-1410^{\circ}\right)$

Answer 7:

It is known that the values of $\operatorname{cosec} x$ repeat after an interval of 2π or 360°.

$$\therefore \operatorname{cosec}\left(-1410^{\circ}\right)=\operatorname{cosec}\left(-1410^{\circ}+4\times 360^{\circ}\right)$$

$$=\operatorname{cosec}\left(-1410^{\circ}+1440^{\circ}\right)$$

$$=\operatorname{cosec} 30^{\circ}=2$$

Question 8:

Find the value of the trigonometric function $\tan \frac{19\pi}{3}$

Answer 8:

It is known that the values of $\tan x$ repeat after an interval of π or 180°.

$$\therefore \tan \frac{19\pi}{3}$$

$$= \tan 6\frac{1}{3}\pi = \tan\left(6\pi + \frac{\pi}{3}\right)$$

$$= \tan \frac{\pi}{3} = \tan 60^{\circ} = \sqrt{3}$$

Question 9:

Find the value of the trigonometric function $\sin\left(-\frac{11\pi}{3}\right)$

Answer 9:

It is known that the values of $\sin x$ repeat after an interval of 2Π or 360°

$$\therefore \sin\left(-\frac{11\pi}{3}\right)$$

Substitute the values

$$= \sin\left(-\frac{11\pi}{3} + 2\times 2\pi\right)$$

Simplify the term

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Question 10:

Find the value of the trigonometric function $\cot\left(-\frac{15\pi}{4}\right)$

Answer 10:

It is known that the values of $\cot x$ repeat after an interval of Π or 180°.

Given that

$$\therefore \cot\left(-\frac{15\pi}{4}\right)$$

Substitute the values

$$=\cot\left(-\frac{15\pi}{4}+4\pi\right)$$

Simplify the term

$$=\cot\frac{\pi}{4}=1$$

Example 10

Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}=1$$

Solution

Given that

$$\text{L.H.S.}\quad =3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

Substituting the terms

$$=3\times\frac{1}{2}\times 2-4\sin\left(\pi-\frac{\pi}{6}\right)\times 1$$

$$=3-4\sin\frac{\pi}{6}$$

$$=3-4\times\frac{1}{2}=1=\text{ R.H.S.}$$

Hence proved

Example 11

Find the value of $\sin 15^\circ$.

Solution

We have

$\sin 15^\circ = \sin\left(45^\circ - 30^\circ\right)$

Expand terms

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

Substitute value

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example 12

Find the value of $\tan \frac{13\pi}{12}$.

Solution

We have

$$\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right)$$

Expand terms

$$= \tan \frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

simplify

$$= \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= 2 - \sqrt{3}$$

Example 13

Prove that

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}$$

Solution

We have

$$\text{L.H.S.} =\frac{\sin(x+y)}{\sin(x-y)}=\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$$

Dividing the numerator and denominator by $\cos x\cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}$$

Example 14

Show that

$$\tan 3x\tan 2x\tan x=\tan 3x-\tan 2x-\tan x$$

Solution

We know that $3x=2x+x$

Therefore, $\tan 3x=\tan(2x+x)$

or $\quad \tan 3x=\dfrac{\tan 2x+\tan x}{1-\tan 2x\tan x}$

or $\quad \tan 3x-\tan 3x\tan 2x\tan x=\tan 2x+\tan x$

or $\quad \tan 3x-\tan 2x-\tan x=\tan 3x\tan 2x\tan x$ or

$\tan 3x\tan 2x\tan x=\tan 3x-\tan 2x-\tan x$.

Example 15

Prove that

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$$

Solution 15

Given that

L.H.S. $=\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

Expand terms

$$=2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-\left(\frac{\pi}{4}-x\right)}{2}\right)$$

simplify

$$=2\cos\frac{\pi}{4}\cos x=2\times\frac{1}{\sqrt{2}}\cos x=\sqrt{2}\cos x=\text{ R.H.S.}$$

Example 16

Prove that $\frac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x}=\cot x$

Solution

Given that

$$\text{L.H.S. }=\frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}}$$

simplify

$$=\frac{\cos x}{\sin x}$$

$$=\cot x$$

$$=\text{ R.H.S.}$$

Example 17

Prove that $=\frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$

Solution

Given that

L.H.S. $= \frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x}$

$= \frac{\sin 5x + \sin x - 2\sin 3x}{\cos 5x - \cos x}$

simplify

$= \frac{2\sin 3x\cos 2x - 2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x(\cos 2x - 1)}{\sin 3x\sin 2x}$

$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x =$ R.H.S.

Exercise 3.3

Question 1:

$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$

Answer 1:

Given that

L.H.S. $= \sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4}$

Put the values

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2$

$= \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2}$

$=$ R.H.S.

Question 2:

Prove that $2\sin^2\frac{\pi}{6} + \text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

Answer 2:

Given that

L.H.S. $= 2\sin^2\frac{\pi}{6} + \mathrm{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}$

$= 2\left(\frac{1}{2}\right)^2 + \mathrm{cosec}^2\left(\pi + \frac{\pi}{6}\right)\left(\frac{1}{2}\right)^2$

$= 2\times\frac{1}{4} + \left(-\mathrm{cosec}\frac{\pi}{6}\right)^2\left(\frac{1}{4}\right)$

$= \frac{1}{2} + (-2)^2\left(\frac{1}{4}\right)$

$= \frac{1}{2} + \frac{4}{4} = \frac{1}{2} + 1 = \frac{3}{2}$

$=$ R.H.S.

Question 3:

Prove that $\cot^2\frac{\pi}{6} + \mathrm{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

Answer 3:

Given that

L.H.S. $= \cot^2\frac{\pi}{6} + \mathrm{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6}$

$= (\sqrt{3})^2 + \mathrm{cosec}\left(\pi - \frac{\pi}{6}\right) + 3\left(\frac{1}{\sqrt{3}}\right)^2$

$= 3 + \mathrm{cosec}\frac{\pi}{6} + 3\times\frac{1}{3}$

$= 3 + 2 + 1 = 6$

$=$ R.H.S

Question 4:

Prove that $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

Answer 4:

Given that

$$\text{L.H.S} = 2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}$$

$$=2\left\{\sin\left(\pi-\frac{\pi}{4}\right)\right\}^2+2\left(\frac{1}{\sqrt{2}}\right)^2+2(2)^2$$

$$=2\left\{\sin\frac{\pi}{4}\right\}^2+2\times\frac{1}{2}+8$$

$$=2\left(\frac{1}{\sqrt{2}}\right)^2+1+8$$

$$=1+1+8$$

$$=10$$

$$=\text{R.H.S}$$

Question 5:

Find the value of:

(i) $\sin 75^\circ$

(ii) $\tan 15^\circ$

Answer 5:

(i) $\sin 75^\circ = \sin\left(45^\circ+30^\circ\right)$

$$=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$

$$[\sin(x+y)=\sin x\cos y+\cos x\sin y]$$

$$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\tan 15^\circ=\tan\left(45^\circ-30^\circ\right)$

$$=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}\left[\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}\right]$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+1\left(\frac{1}{\sqrt{3}}\right)}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}=\frac{3+1-2\sqrt{3}}{(\sqrt{3})^2-(1)^2}$$

$$=\frac{4-2\sqrt{3}}{3-1}=2-\sqrt{3}$$

Question 6:

Prove that: $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$

Answer 6:

Given that

$$\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)$$

$$=\frac{1}{2}\left[2\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)\right]+\frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)\right]$$

$$=\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}+\cos\left\{\left(\frac{\pi}{4}-x\right)-\left(\frac{\pi}{4}-y\right)\right\}\right]$$

$$=\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}-\cos\left\{\left(\frac{\pi}{4}-x\right)-\left(\frac{\pi}{4}-y\right)\right\}\right]$$

$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$

$-2\sin A\sin B=\cos(A+B)-\cos(A-B)]$

$$=2\times\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}\right]$$

$$=\cos\left[\frac{\pi}{2}-(x+y)\right]$$

$$=\sin(x+y)$$

$$=R\cdot H\cdot S$$

Question 7:

Prove that: $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

Answer 7:

It is known that

$$t\,\text{an}(\text{A}+\text{B})=\frac{\tan \text{A}+\tan \text{B}}{1-\tan \text{A}\tan \text{B}}$$

$$\tan(\text{A}-\text{B})=\frac{\tan \text{A}-\tan \text{B}}{1+\tan \text{A}\tan \text{B}}$$

Then just apply the formula

We get: $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

proved

Question 8:

$\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\dfrac{\pi}{2}+x\right)}=\cot^2 x$ Prove that $\dfrac{x}{2}$

Answer 8:

Given that

$$\text{L.H.S.}\quad =\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\dfrac{\pi}{2}+x\right)}$$

$$=\frac{[-\cos x][\cos x]}{(\sin x)(-\sin x)}$$

$$=\frac{-\cos^2 x}{-\sin^2 x}$$

$$=\cot^2 x$$

$= \text{R.H.S.}$

Question 9:

$$\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$$

Answer 9:

Given that

$$\text{L.H.S.} \quad =\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]$$

$$=\sin x\cos x[\tan x+\cot x]$$

$$=\sin x\cos x\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)$$

$$=(\sin x\cos x)\left[\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right]$$

$$=1= \text{R.H.S.}$$

Question 10:

Prove that $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$

Answer 10:

Given that

$$\text{L.H.S.} \ =\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x$$

$$=\frac{1}{2}[2\sin(n+1)x\sin(n+2)x+2\cos(n+1)x\cos(n+2)x]$$

$$=\frac{1}{2}[\cos\{(n+1)x-(n+2)x\}-\cos\{(n+1)x+(n+2)x\}]$$

$$\because -2\cos\{(n+1)x+(n+2)x\}+\cos\{(n+1)x-(n+2)x\}]$$

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

$$=\frac{1}{2}\times 2\cos\{(n+1)x-(n+2)x\}$$

$$=\cos(-x)=\cos x= \text{R.H.S.}$$

Question 11:

Prove that $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$

Answer 11:

Given that

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{ L.H.S. } =\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)$$

$$=-2\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)+\left(\frac{3\pi}{4}-x\right)}{2}\right\}\cdot\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)-\left(\frac{3\pi}{4}-x\right)}{2}\right\}$$

simplify

$$=-2\sin\left(\frac{3\pi}{4}\right)\sin x$$

$$=-2\sin\left(\pi-\frac{\pi}{4}\right)\sin x$$

$$=-2\sin\frac{\pi}{4}\sin x$$

$$=-2\times\frac{1}{\sqrt{2}}\times\sin x$$

$$=-\sqrt{2}\sin x$$

$$= \text{R.H.S.}$$

Question 12:

Prove that $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$

Answer 12:

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$\text{L.H.S.} = \sin^2 6x - \sin^2 4x$

$= (\sin 6x + \sin 4x)(\sin 6x - \sin 4x)$

$$= \left[2\sin\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right)\right]\left[2\cos\left(\frac{6x+4x}{2}\right)\cdot\sin\left(\frac{6x-4x}{2}\right)\right]$$

$= (2\sin 5x \cos x)(2\cos 5x$

$\sin x) = (2\sin 5x \cos 5x)(2$

$\sin x \cos x)$

$= \sin 10x \sin 2x$

$= \text{R.H.S.}$

Question 13:

Prove that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

Answer 13:

It is known that

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$\therefore\ \text{L.H.S.} = \cos^2 2x - \cos^2 6x$

$= (\cos 2x + \cos 6x)(\cos 2x - 6x)$

$$= \left[2\cos\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)\right]\left[-2\sin\left(\frac{2x+6x}{2}\right)\sin\frac{(2x-6x)}{2}\right]$$

$= [2\cos 4x \cos(-2x)][-2\sin 4x \sin(-2x)]$

$= [2\cos 4x \cos 2x][-2\sin 4x(-\sin 2x)]$

$= (2\sin 4x \cos 4x)(2\sin 2x \cos 2x)$

$= \sin 8x \sin 4x = \text{R.H.S.}$

Question 14:

Prove that $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

Answer 14:

$\text{L.H.S.} = \sin 2x + 2\sin 4x + \sin 6x$

$$=[\sin 2x+\sin 6x]+2\sin 4x$$

$$=\left[2\sin\left(\frac{2x+6x}{2}\right)\left(\frac{2x-6x}{2}\right)\right]+2\sin 4x$$

$$\left[\because \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$=2\sin 4x\cos(-2x)+2\sin 4x$$

$$=2\sin 4x\cos 2x+2\sin 4x$$

$$=2\sin 4x(\cos 2x+1)$$

$$=2\sin 4x\left(2\cos^2 x-1+1\right)$$

$$=2\sin 4x\left(2\cos^2 x\right)$$

$$=4\cos^2 x\sin$$

$4x=$ R.H.S.

Question 15:

Prove that $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$

Answer 15:

Given that

L.H.S $=\cot 4x(\sin 5x+\sin 3x)$

$$=\frac{\cot 4x}{\sin 4x}\left[2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)\right]$$

$$\left[\because \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$=\left(\frac{\cos 4x}{\sin 4x}\right)[2\sin 4x\cos x]$$

$$=2\cos 4x\cos x$$

R.H.S. $=\cot x(\sin 5x-\sin 3x)$

$$=\frac{\cos x}{\sin x}\left[2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)\right]$$

$$\left[\because \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$=\frac{\cos x}{\sin x}[2\cos 4x\sin x]$$

$$=2\cos 4x\cdot\cos x$$

L.H.S. = R.H.S.

Question 16:

Prove that $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$

Answer 16:

It is known that

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right), \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{ L.H.S } =\frac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}$$

$$=\frac{-2\sin\left(\frac{9x+5x}{2}\right)\cdot\sin\left(\frac{9x-5x}{2}\right)}{2\cos\left(\frac{17x+3x}{2}\right)\cdot\sin\left(\frac{17x-3x}{2}\right)}$$

$$=\frac{-2\sin 7x\cdot\sin 2x}{2\cos 10x\cdot\sin 7x}$$

$$=-\frac{\sin 2x}{\cos 10x}$$

$$=R\cdot H\cdot S$$

Question 17:

Prove that: $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

Answer 17:

It is known that

$$\sin \mathrm{A}+\sin \mathrm{B}=2\sin\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right)\cos\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right), \cos \mathrm{A}+\cos \mathrm{B}=2\cos\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right)\cos\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right)$$

$\therefore$ L.H.S. $=\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}$

$$=\frac{2\sin\left(\frac{5x+3x}{2}\right)\cdot\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right)\cdot\cos\left(\frac{5x-3x}{2}\right)}$$

$$=\frac{2\sin 4x\cdot\cos x}{2\cos 4x\cdot\cos x}$$

$$=\frac{\sin 4x}{\cos 4x}$$

$=\tan 4x=$ R.H.S.

Question 18:

Prove that $\dfrac{\sin x-\sin y}{\cos x+\cos y}=\tan\dfrac{x-y}{2}$

Answer 18:

It is known that

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right), \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$\therefore$ L.H.S. $=\dfrac{\sin x-\sin y}{\cos x+\cos y}$

$$=\frac{2\cos\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)}$$

$$=\frac{\sin\left(\frac{x-y}{2}\right)}{\cos\left(\frac{x-y}{2}\right)}$$

$$=\tan\left(\frac{x-y}{2}\right)=R.H.S$$

Question 19:

Prove that $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$

Answer 19:

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\therefore \text{ L.H.S. }=\frac{\sin x+\sin 3x}{\cos x+\cos 3x}$$

$$=\frac{2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$=\frac{\sin 2x}{\cos 2x}$$

$$=\tan 2x$$

$$=\text{ R.H.S}$$

Question 20:

Prove that $\dfrac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}=2\sin x$

Answer 20:

It is known that

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),\cos^2 A-\sin^2 A=\cos 2A$$

$$\therefore \text{ L.H.S. }=\frac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}$$

$$=\frac{2\cos\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}{-\cos 2x}$$

$$=\frac{2\cos 2x\sin(-x)}{-\cos 2x}$$

$$=-2\times(-\sin x)$$

$$=2\sin x=\text{ R.H.S.}$$

Question 21:

Prove that $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

Solution 21:

Given that

$$\text{L.H.S.} =\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}$$

$$=\frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\sin 3x}$$

$$=\frac{(\cos 4x+\cos 2x)+\cos 3x}{(\sin 4x+\sin 2x)+\sin 3x}$$

$$\left[\because \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

Apply the formula

$$=\frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x}$$

$$=\frac{\cos 3x(2\cos x+1)}{\sin 3x(2\cos x+1)}$$

$$=\cot 3x=R\cdot H\cdot S$$

Question 22:

Prove that $\cot x\cot 2x-\cot 2x\cot 3x-\cot 3x\cot x=1$

Answer 22:

Given that

$$\text{L.H.S.} =\cot x\cot 2x-\cot 2x\cot 3x-\cot 3x\cot x$$

$$=\cot x\cot 2x-\cot 3x(\cot 2x+\cot x)$$

$$=\cot x\cot 2x-\cot(2x+x)(\cot 2x+\cot x)$$

$$=\cot x\cot 2x-\left[\frac{\cot 2x\cot x-1}{\cot x+\cot 2x}\right](\cot 2x+\cot x)$$

$\left[\because \cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}\right]$

$=\cot x\cot 2x-(\cot 2x\cot x-1)=1=$ R.H.S.

Question 23:

Prove that $\tan 4x=\frac{4\tan x\left(1-\tan^2 x\right)}{1-6\tan^2 x+\tan^4 x}$

Answer 23:

It is known that. $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$

$\therefore$ L.H.S. $=\tan 4x=\tan 2(2x)$

$$=\frac{2\tan 2x}{1-\tan^2(2x)}$$

$$=\frac{2\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}$$

$$=\frac{4\tan x}{1-\tan^2 x}\Bigg)$$

$$=\frac{4-\frac{1\tan^2 x}{\left(1-\tan^2 x\right)^2}\Bigg]}{\left(\frac{4\tan x}{1-\tan^2 x}\right)}$$

$$=\frac{\left(1-\tan^2 x\right)^2-4\tan^2 x\Big]}{\left(1-\tan^2 x\right)^2}$$

$$=\frac{4\tan x\left(1-\tan^2 x\right)}{1-\tan^2 x\Big)^2-4\tan^2 x}$$

$$= \frac{4\tan x\left(1-\tan^2 x\right)}{1-6\tan^2 x+\tan^4 x} = \text{ R.H.S.}$$

Question 24:

Prove that: $\cos 4x = 1-8\sin^2 x\cos^2 x$

Answer 24:

$\text{L.H.S. } = \cos 4x$

$= \cos 2(2x)$

$= 1-2\sin^2 2x\left[\cos 2A = 1-2\sin^2 A\right]$

$= 1-2(2\sin x\cos x)^2[\sin 2A = 2\sin A\cos A]$

$= 1-8\sin^2 x$

$\cos^2 x = \text{ R.H.S.}$

Question 25:

Prove that: $\cos 6x = 32\cos^6 x-48\cos^4 x+18\cos^2 x-1$

Answer 25:

$\text{L.H.S. } = \cos 6x$

$= \cos 3(2x)$

$= 4\cos^3 2x-3\cos 2x\left[\cos 3A = 4\cos^3 A-3\cos A\right]$

$= 4\left[\left(2\cos^2 x-1\right)^3-3\left(2\cos^2 x-1\right)\left[\cos 2x = 2\cos^2 x-1\right]\right.$

$= 4\left[\left(2\cos^2 x\right)^3-(1)^3-3\left(2\cos^2 x\right)^2+3\left(2\cos^2 x\right)\right]-6\cos^2 x+3$

$= 4\left[8\cos^6 x-1-12\cos^4 x+6\cos^2 x\right]-6\cos^2 x+3$

$= 32\cos^6 x-4-48\cos^4 x+24\cos^2 x-6\cos^2 x+3$

$= 32\cos^6 x-48\cos^4 x+18$

$\cos^2 x-1 = \text{ R.H.S.}$

Example 18

Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$

Solution

Given that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \text{ and}$$

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$$

simplify

$$= \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}.$$

Value of $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Example 19

Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.

Solution

We know that, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ and

$$\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

Thus $\tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$.

Therefore, $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

Already we seen that

$\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n+1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$

Example 20.

Find the solution of $\sin x=-\frac{\sqrt{3}}{2}$.

Solution

Given that

$$\sin x=-\frac{\sqrt{3}}{2}$$

$$=-\sin\frac{\pi}{3}$$

$$=\sin\left(\pi+\frac{\pi}{3}\right)$$

$$=\sin\frac{4\pi}{3}$$

Hence $\sin x=\sin\frac{4\pi}{3}$,

which gives $x=n\pi+(-1)^n\frac{4\pi}{3}$,

where $n\in\mathbf{Z}$

Example 21

Solve $\cos x=\frac{1}{2}$.

Solution

Given that

$$\cos x=\frac{1}{2}$$

$$=\cos\frac{\pi}{3}$$

Therefore

$$x=2n\pi\pm\frac{\pi}{3},$$

where $n \in \mathbf{Z}$.

Example 22

Solve $\tan 2x = -\cot\left(x+\frac{\pi}{3}\right)$.

Solution

Given that

$$\tan 2x = -\cot\left(x+\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2}+x+\frac{\pi}{3}\right)$$

or

$$\tan 2x = \tan\left(x+\frac{5\pi}{6}\right)$$

Hence

$$2x = n\pi + x + \frac{5\pi}{6},$$

where $n \in \mathbf{Z}$

or $x = n\pi + \frac{5\pi}{6}$,

where $n \in \mathbf{Z}$

Example 23

Solve $\sin 2x - \sin 4x + \sin 6x = 0$

Solution

Given that

$$\sin 2x - \sin 4x + \sin 6x = 0$$

Rewrite

$$\sin 6x + \sin 2x - \sin 4x = 0$$

$$2\sin 4x \cos 2x - \sin 4x = 0$$

$$\sin 4x(2\cos 2x - 1) = 0$$

i.e.

Therefore $\sin 4x = 0$ or $\cos 2x = \frac{1}{2}$

i.e. $\sin 4x = 0$ or $\cos 2x = \cos\frac{\pi}{3}$

Hence $4x = n\pi$ or $2x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

i.e. $x = \frac{n\pi}{4}$ or $x = n\pi \pm \frac{\pi}{6}$, where $n \in \mathbf{Z}$.

Example 24

Solve $2\cos^2 x + 3\sin x = 0$

Solution

Given that

$2\cos^2 x + 3\sin x = 0$

$2\left(1 - \sin^2 x\right) + 3\sin x = 0$

or

$2\sin^2 x - 3\sin x - 2 = 0$

or

$(2\sin x + 1)(\sin x - 2) = 0$

Hence $\sin x = -\frac{1}{2}$ or $\sin x = 2$

But $\sin x = 2$ is not possible

Then

$\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$

Hence,

$x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in \mathbf{Z}$.

Exercise 3.4

Question 1:

Find the principal and general solutions of the equation $\tan x = \sqrt{3}$

Answer 1:

Given that

$\tan x = \sqrt{3}$

It is known that $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

Therefore, the principal solutions are $x = \frac{\pi}{3}$ and $\frac{4\pi}{3}$.

Now, $\tan x = \tan\frac{\pi}{3}$

$\Rightarrow x = n\pi + \frac{\pi}{3}$, where $n \in Z$

Therefore, the general solution is $x = n\pi + \frac{\pi}{3}$, where $n \in Z$

Question 2:

Find the principal and general solutions of the equation $\sec x = 2$

Answer 2:

Given that

$\sec x = 2$

It is known that $\sec\frac{\pi}{3} = 2$ and $\sec\frac{5\pi}{3} = \sec\left(2\pi - \frac{\pi}{3}\right) = \sec\frac{\pi}{3} = 2$

Therefore, the principal solutions are $x = ^{\frac{\pi}{3}}$ and $\frac{5\pi}{3}$.

Now, $\sec x = \sec\frac{\pi}{3}$

$\Rightarrow \cos x = \cos\frac{\pi}{3}$ $\left[\sec x = \frac{1}{\cos x}\right]$

$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}$, where $n \in Z$

Therefore, the general solution is $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

Question 3:

Find the principal and general solutions of the equation $\cot x = -\sqrt{3}$

Answer 3:

Given that

$\cot x = -\sqrt{3}$

It is known that $\cot\frac{\pi}{6} = \sqrt{3}$

$\therefore \cot\left(\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$ and $\cot\left(2\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$

i.e., $\cot\frac{5\pi}{6} = -\sqrt{3}$ and $\cot\frac{11\pi}{6} = -\sqrt{3}$

Therefore, the principal solutions are $x = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

Now, $\cot x = \cot\frac{5\pi}{6}$

$\Rightarrow \tan x = \tan\frac{5\pi}{6}$ $\left[\cot x = \frac{1}{\tan x}\right]$

$\Rightarrow x = n\pi + \frac{5\pi}{6}$, where $n \in Z$

Therefore, the general solution is $x = n\pi + \frac{5\pi}{6}$, where $n \in Z$

Question 4:

Find the general solution of $\operatorname{cosec} x = -2$

Answer 4:

Given that

$\operatorname{cosec} x = -2$

It is known that

$\operatorname{cosec}\frac{\pi}{6} = 2$

$\therefore \operatorname{cosec}\left(\pi + \frac{\pi}{6}\right) = -\operatorname{cosec}\frac{\pi}{6} = -2$ and $\operatorname{cosec}\left(2\pi - \frac{\pi}{6}\right) = -\operatorname{cosec}\frac{\pi}{6} = -2$

i.e., $\operatorname{cosec}\frac{7\pi}{6} = -2$ and $\operatorname{cosec}\frac{11\pi}{6} = -2$

Therefore, the principal solutions are $x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Now, $\operatorname{cosec} x = \operatorname{cosec}\frac{7\pi}{6}$

$\Rightarrow \sin x = \sin\frac{7\pi}{6} \quad \left[\operatorname{cosec} x = \frac{1}{\sin x}\right]$

$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in Z$

Therefore, the general solution is $x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in Z$

Question 5:

Find the general solution of the equation $\cos 4x = \cos 2x$

Answer 5:

Given that

$\cos 4x = \cos 2x$

$\Rightarrow \cos 4x - \cos 2x = 0$

$\Rightarrow -2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right) = 0$

$\left[\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$

$\Rightarrow \sin 3x \sin x = 0$

$\Rightarrow \sin 3x = 0 \quad \text{or} \quad \sin x = 0$

$\therefore 3x = n\pi \quad \text{or} \quad x = n\pi, \text{ where } n \in Z$

$\Rightarrow x = \frac{n\pi}{3} \quad \text{or} \quad x = n\pi, \text{ where } n \in Z$

Question 6:

Find the general solution of the equation $\cos 3x + \cos x - \cos 2x = 0$

Answer 6:

Given that

$\cos 3x + \cos x - \cos 2x = 0$

$\Rightarrow 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) - \cos 2x = 0 \quad \left[\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$

$\Rightarrow 2\cos 2x \cos x - \cos 2x = 0 \Rightarrow \cos 2x(2\cos x - 1) = 0 \Rightarrow \cos 2x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$

$\Rightarrow \cos 2x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$

$\therefore 2x = (2n+1)\frac{\pi}{2} \quad \text{or} \quad \cos x = \cos\frac{\pi}{3}, \text{ where } n \in Z$

$\Rightarrow x = (2n+1)\frac{\pi}{4} \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$

Question 7:

Find the general solution of the equation $\sin 2x + \cos x = 0$

Answer 7:

Given that

$\sin 2x + \cos x = 0$

$\Rightarrow 2\sin x \cos x + \cos x = 0$

$\Rightarrow \cos x(2\sin x + 1) = 0$

$\Rightarrow \cos x = 0 \quad \text{or} \quad 2\sin x + 1 = 0$

Now, $\cos x = 0 \Rightarrow \cos x = (2n+1)\frac{\pi}{2}$, where $n \in Z$

$2\sin x + 1 = 0$

$$\Rightarrow \sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in Z$$

Therefore, the general solution is $(2n+1)\frac{\pi}{2}$ or $n\pi + (-1)^n \frac{7\pi}{6}, n \in Z$

Question 8:

Find the general solution of the equation $\sec^2 2x = 1 - \tan 2x$

Answer 8:

Given that

$$\sec^2 2x = 1 - \tan 2x$$

$$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$$

$$\Rightarrow \tan^2 2x + \tan 2x = 0$$

$$\Rightarrow \tan 2x(\tan 2x + 1) = 0$$

$$\Rightarrow \tan 2x = 0 \quad \text{or} \quad \tan 2x + 1 = 0$$

Now, $\tan 2x = 0$

$$\Rightarrow \tan 2x = \tan 0$$

$$\Rightarrow 2x = n\pi + 0, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2}, \text{ where } n \in Z$$

$$\tan 2x + 1 = 0$$

$$\Rightarrow \tan 2x = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$$

$$\Rightarrow 2x = n\pi + \frac{3\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2} + \frac{3\pi}{8}, \text{ where } n \in Z$$

Therefore, $\frac{n\pi}{2}$ or $\frac{n\pi}{2} + \frac{3\pi}{8}, n \in \mathbf{Z}$

Question 9:

Find the general solution of the equation $\sin x + \sin 3x + \sin 5x = 0$

Answer 9:

Given that

$$\sin x + \sin 3x + \sin 5x = 0$$

$$(\sin x + \sin 5x) + \sin 3x = 0$$

$$\Rightarrow \left[2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right)\right] + \sin 3x = 0 \quad \left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$\Rightarrow 2\sin 3x \cos(-2x) + \sin 3x = 0$$

$$\Rightarrow 2\sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \sin 3x(2\cos 2x + 1) = 0$$

$$\Rightarrow \sin 3x = 0 \quad \text{or} \quad 2\cos 2x + 1 = 0$$

Now, $\sin 3x = 0 \Rightarrow 3x = n\pi$, where $n \in Z$

i.e., $x = \frac{n\pi}{3}$, where $n \in Z$

$$2\cos 2x + 1 = 0$$

$$\Rightarrow \cos 2x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \cos 2x = \cos\frac{2\pi}{3}$$

$$\Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3}, \text{ where } n \in Z$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$$

Therefore $\frac{n\pi}{3}$ or $n\pi \pm \frac{\pi}{3}, n \in Z$

Miscellaneous Examples

Example 25

If $\sin x=\frac{3}{5},\cos y=-\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin(x+y)$.

Solution

We know that

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

Now

Therefore $\cos x=\pm\frac{4}{5}$

Since x lies in 2^{nd} quadrant,

$\cos x$ is negative.

Hence $\cos x=-\frac{4}{5}$

Now $\sin^2 y=1-\cos^2 y=1-\frac{144}{169}=\frac{25}{169}$

i.e. $\sin y=\pm\frac{5}{13}$.

Since y lies in second quadrant, hence sin y is positive.

Therefore, $\sin y=\frac{5}{13}$. Substituting the values of $\sin x,\sin y,\cos x$ and $\cos y$ in (1), we get

$\sin(x+y)=\frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13}$

$=-\frac{36}{65}-\frac{20}{65}$

$=-\frac{56}{65}$

Example 26

Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}$$

Solution

We have

$$\text{L.H.S.} \quad = \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right) + \cos\left(2x-\frac{x}{2}\right) - \cos\left(\frac{9x}{2}+3x\right) - \cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2} + \cos\frac{3x}{2} - \cos\frac{15x}{2} - \cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2} - \cos\frac{15x}{2}\right]$$

$$= \left[\frac{1}{2} - 2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

simplify

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{ R.H.S.}$$

Example 27

Find the value of $\tan\frac{\pi}{8}$.

Solution

Given that

Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$

or

$$\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

Let $y=\tan\frac{\pi}{8}$. Then $1=\frac{2y}{1-y^2}$

or

$$y^2+2y-1=0$$

Therefore

$$y=\frac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}$$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y=\tan\frac{\pi}{8}$ is positve.

Hence

$$\tan\frac{\pi}{8}=\sqrt{2}-1$$

Example 28

If $\tan x=\frac{3}{4}, \pi<x<\frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution

Given that

Since $\pi<x<\frac{3\pi}{2}, \cos x$ is negative.

Also

$$\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}$$

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now

$$\sec^2 x=1+\tan^2 x=1+\frac{9}{16}=\frac{25}{16}$$

Therefore

$$2\sin^2\frac{x}{2}=1-\cos x=1+\frac{4}{5}=\frac{9}{5}.$$

Therefore

$\sin^2\frac{x}{2}=\frac{9}{10}$

or

$\sin\frac{x}{2}=\frac{3}{\sqrt{10}}$

Again

$2\cos^2\frac{x}{2}=1+\cos x=1-\frac{4}{5}=\frac{1}{5}$

Therefore

$\cos^2\frac{x}{2}=\frac{1}{10}$

or

$\cos\frac{x}{2}=-\frac{1}{\sqrt{10}}$ (Why?)

Hence $\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{3}{\sqrt{10}}\times\left(\frac{-\sqrt{10}}{1}\right)=-3.$

Example 29

Prove that $\cos^2 x+\cos^2\left(x+\frac{\pi}{3}\right)+\cos^2\left(x-\frac{\pi}{3}\right)=\frac{3}{2}$.

Solution

We have

$$\text{L.H.S.}=\frac{1+\cos 2x}{2}+\frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2}+\frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}$$

$$=\frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$=\frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$=\frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$=\frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$=\frac{1}{2}[3+\cos 2x-\cos 2x]=\frac{3}{2}=\text{ R.H.S.}$$

Miscellaneous Exercise on chapter 3

Question 1:

Prove that: $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Answer 1:

Given that

L.H.S.

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\left(\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}\right)\cos\left(\frac{3\pi}{13}-\frac{5\pi}{13}\right)\left[\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

simplify

$$=2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right]$$

$$= 2\cos\frac{\pi}{13}\left[2\cos\left(\frac{\frac{9\pi}{13}+\frac{4\pi}{13}}{2}\right)\cos\left(\frac{9\pi}{13}-\frac{4\pi}{13}\right)\right]$$

$$= 2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right]$$

$$= 2\cos\frac{\pi}{13}\times 2\times 0\times\cos\frac{5\pi}{26}$$

$= 0 =$ R.H.S

Question 2:

Prove that: $(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$

Answer 2:

Given that

L.H.S.

$$= (\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x$$

$$= \sin 3x\sin x+\sin^2 x+\cos 3x\cos x-\cos^2 x$$

$$= \cos 3x\cos x+\sin 3x\sin x-\left(\cos^2 x-\sin^2 x\right)$$

$$= \cos(3x-x)-\cos 2x \quad [\cos(A-B)=\cos A\cos B+\sin A\sin B]$$

$$= \cos 2x-\cos 2x$$

$= 0$

$=$ RH.S.

Question 3:

Prove that: $(\cos x+\cos y)^2+(\sin x-\sin y)^2=4\cos^2\frac{x+y}{2}$

Answer 3:

Given that

L.H.S. $=(\cos x+\cos y)^2+(\sin x-\sin y)^2$

$$= \cos^2 x+\cos^2 y+2\cos x\cos y+\sin^2 x+\sin^2 y-2\sin x\sin y$$

$$=\left(\cos^2 x+\sin^2 x\right)+\left(\cos^2 y+\sin^2 y\right)+2(\cos x\cos y-\sin x\sin y)$$

$$=1+1+2\cos(x+y) \quad [\cos(A+B)=(\cos A\cos B-\sin A\sin B)]$$

$$=2+2\cos(x+y)$$

$$=2[1+\cos(x+y)]$$

$$=2\left[1+2\cos^2\left(\frac{x+y}{2}\right)-1\right] \quad \left[\cos 2A=2\cos^2 A-1\right]$$

$$=4\cos^2\left(\frac{x+y}{2}\right)= \text{R.H.S.}$$

Question 4:

Prove that: $(\cos x-\cos y)^2+(\sin x-\sin y)^2=4\sin^2\frac{x-y}{2}$

Answer 4:

Given that

L.H.S.

$$=(\cos x-\cos y)^2+(\sin x-\sin y)^2$$

$$=\cos^2 x+\cos^2 y-2\cos x\cos y+\sin^2 x+\sin^2 y-2\sin x\sin y$$

$$=\left(\cos^2 x+\sin^2 x\right)+\left(\cos^2 y+\sin^2 y\right)-2[\cos x\cos y+\sin x\sin y]$$

$$=1+1-2[\cos(x-y)]$$

$$=2[1-\cos(x-y)]$$

$$=2\left[1-\left\{1-2\sin^2\left(\frac{x-y}{2}\right)\right\}\right] \quad [\cos(A-B)=\cos A\cos B+\sin A\sin B]$$

$$=4\sin^2\left(\frac{x-y}{2}\right)= \text{R.H.S.} \quad \left[\cos 2A=1-2\sin^2 A\right]$$

Question 5:

Prove that: $\sin x+\sin 3x+\sin 5x+\sin 7x=4\cos x\cos 2x\sin 4x$

Answer 5:

Given that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

$$?.\text{ H.S. }=\sin x+\sin 3x+\sin 5x+\sin 7x$$

$$=(\sin x+\sin 5x)+(\sin 3x+\sin 7x)$$

$$=2\sin\left(\frac{x+5x}{2}\right)\cdot\cos\left(\frac{x-5x}{2}\right)+2\sin\left(\frac{3x+7x}{2}\right)\cos\left(\frac{3x-7x}{2}\right)$$

$$=2\sin 3x\cos(-2x)+2\sin 5x\cos(-2x)$$

$$=2\sin 3x\cos 2x+2\sin 5x\cos 2x$$

$$=2\cos 2x[\sin 3x+\sin 5x]$$

$$=2\cos 2x\left[2\sin\left(\frac{3x+5x}{2}\right)\cdot\cos\left(\frac{3x-5x}{2}\right)\right]$$

$$=2\cos 2x[2\sin 4x\cdot\cos(-x)]$$

$$=4\cos 2x\sin 4x\cos x=\text{ R.H.S.}$$

Question 6:

Prove that: $\dfrac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}=\tan 6x$

Answer 6:

We have

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right),\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

Given that

$$\text{L.H.S. }=\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(2\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}$$

$$=\frac{[7x+5x}{2}\Big)\cdot\cos\left(\frac{7x-5x}{2}\right)\Big]\Big]+\left[2\sin\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]$$

$$\left[2\cos\left(\frac{7x+5x}{2}\right)\cdot\cos\left(\frac{7x-5x}{2}\right)\right]+\left[2\cos\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]$$

$$=\frac{[2\sin 6x\cdot\cos x]+[2\sin 6x\cdot\cos 3x]}{[2\cos 6x\cdot\cos x]+[2\cos 6x\cdot\cos 3x]}$$

$$= \frac{2\sin 6x[\cos x + \cos 3x]}{2\cos 6x[\cos x + \cos 3x]}$$

$$= \tan 6x$$

$$= \text{ R.H.S.}$$

Question 7:

Prove that: $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\frac{x}{2}\cos\frac{3x}{2}$

Answer 7:

Given that

$$\text{L.H.S.} = \sin 3x + \sin 2x - \sin x$$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 3x + \left[2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)\right]\left[\sin A - \sin B = 2cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \sin 3x + \left[2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)\right]$$

$$= \sin 3x + 2\cos\frac{3x}{2}\sin\frac{x}{2}$$

$$= 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2} + 2\cos\frac{3x}{2}\sin\frac{x}{2}$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[2\sin\{\frac{\sin 2A = 2\sin A\cdot\cos B]}{2}\left[\left(\frac{3x}{2}\right)+\left(\frac{x}{2}\right)\right]\cos\left\{\frac{\left(\frac{3x}{2}\right)-\left(\frac{x}{2}\right)}{2}\right\}]\right.$$

$$\left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right)\cdot 2\sin x\cos\left(\frac{x}{2}\right)$$

$$= 4\sin x\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) = \text{ R.HS.}$$

Question 8:

Find $\sin x/2, \cos x/2$ and $\tan x/2$, if $\tan x = -\frac{4}{3}, x$ in quadrant II

Answer 8:

Here, x is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

Therefore, $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are lies in first quadrant.

It is given that $\tan x = -\frac{4}{3}$

$\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{-4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$

$\therefore \cos^2 x = \frac{9}{25}$

$\Rightarrow \cos x = \pm\frac{3}{5}$

As x is in quadrant II, $\cos x$ is negative.

$\cos x = \frac{-3}{5}$

Now, $\cos x = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow \frac{-3}{5} = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow 2\cos^2\frac{x}{2} = 1 - \frac{3}{5}$

$\Rightarrow 2\cos^2\frac{x}{2} = \frac{2}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{5}$

$\Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}}$

$\therefore \cos\frac{x}{2}=\frac{\sqrt{5}}{5}$

$\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1$

$\Rightarrow \sin^2\frac{x}{2}+\left(\frac{1}{\sqrt{5}}\right)^2=1$

$\Rightarrow \sin^2\frac{x}{2}=1-\frac{1}{5}=\frac{4}{5}$

$\Rightarrow \sin\frac{x}{2}=\frac{2}{\sqrt{5}}$

$\therefore \sin\frac{x}{2}=\frac{2\sqrt{5}}{5}$

$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)}=2$

Thus, the respective values of $\sin\frac{x}{2},\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{2\sqrt{5}}{5},\frac{\sqrt{5}}{5}$, and 2

Question 9:

Find, $\sin\frac{x}{2},\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\cos x=-\frac{1}{3}, x$ in quadrant III

Answer 9:

Here, x is in quadrant III.

i.e., $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

Therefore, $\cos\frac{x}{2}$ and

$\tan\frac{x}{2}$ are negative, where $\sin\frac{x}{2}$ as is positive.

It is given that $\cos x=-\frac{1}{3}$.

$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1-\left(-\frac{1}{3}\right)}{2} = \frac{\left(1+\frac{1}{3}\right)}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \quad \left[\because \sin \frac{x}{2} \text{ is positive}\right]$$

Now

$$\cos x = 2\cos^2 \frac{x}{2} - 1$$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

$$\therefore \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin \frac{x}{2}, \cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{\sqrt{6}}{3}, \frac{-\sqrt{3}}{3}$, and $-\sqrt{2}$

Question 10:

Find $\sin \frac{x}{2}, \cos \frac{x}{2}$ and $\tan \frac{x}{2}$ for $\sin x = \frac{1}{4}, x$ in quadrant II

Answer 10:

Here, x is in quadrant II.

i.e., $\frac{\pi}{2}<x<\pi$

$\Rightarrow \frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{2}$

Therefore, $\sin\frac{x}{2}, \cos\frac{x}{2}, \quad \tan\frac{x}{2}$ are all positive.

It is given that $\sin x=\frac{1}{4}$.

$$\cos^2 x=1-\sin^2 x=1-\left(\frac{1}{4}\right)^2=1-\frac{1}{16}=\frac{15}{16}$$

$$\Rightarrow \cos x=-\frac{\sqrt{15}}{4}$$

$$\sin^2\frac{x}{2}=\frac{1-\cos x}{2}=\frac{1-\left(-\frac{\sqrt{15}}{4}\right)}{2}=\frac{4+\sqrt{15}}{8}$$

$$\Rightarrow \sin\frac{x}{2}=\sqrt{\frac{4+\sqrt{15}}{8}}$$

$$=\sqrt{\frac{4+\sqrt{15}}{8}\times\frac{2}{2}}$$

$$=\sqrt{\frac{8+2\sqrt{15}}{16}}$$

$$=\frac{\sqrt{8+2\sqrt{15}}}{4}$$

$$\cos^2\frac{x}{2}=\frac{1+\cos x}{2}=\frac{1+\left(-\frac{\sqrt{15}}{4}\right)}{2}=\frac{4-\sqrt{15}}{8}$$

$$\Rightarrow \cos\frac{x}{2}=\sqrt{\frac{4-\sqrt{15}}{8}}$$

$$=\sqrt{\frac{4-\sqrt{15}}{8}\times\frac{2}{2}}$$

$$=\sqrt{\frac{8-2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8-2\sqrt{15}}}{4}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4}\right)} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}}$$

$$= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}}$$

$$= \sqrt{\frac{(8+2\sqrt{15})^2}{64-60}} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}$$

Thus, the respective values of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$

are $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}$

and $4+\sqrt{15}$