Chapter 4: Principle of Mathematical Induction

Example 1:

For all $n \ge 1$, prove that

$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Solution:

Given statement be $\mathrm{P}(n)$, i.e.,

$$\mathrm{P}(n): 1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

For $n=1$, $\mathrm{P}(1): 1=\frac{1(1+1)(2\times1+1)}{6}=\frac{1\times2\times3}{6}=1$ which is true.

Assume that $\mathrm{P}(k)$ is true for some positive integer k, i.e.,

$$1^2+2^2+3^2+4^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6} \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is also true. Now,

$$\left(1^2+2^2+3^2+4^2+\ldots+k^2\right)+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)\left(2k^2+7k+6\right)}{6}$$

$$=\frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Hence, $P(k+1)$ is true, whenever $\mathrm{P}(k)$ is true.

$\therefore$ From the principle of mathematical induction, the statement $\mathrm{P}(n)$ is true for all natural numbers n.

Example 2:

Prove that $2^n > n$ for all positive integers n.

Solution:

Let $\mathrm{P}(n): 2^n > n$

When $n=1, 2^1>1$.

Hence $\mathrm{P}(1)$ is true.

Assume that $\mathrm{P}(k)$ is true for any positive integer k,

$$2^k>k \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Multiplying both sides of equation (1) by 2,

$$2\times 2^k>2k$$

$$2^{k+1}>2k=k+k>k+1$$

Therefore, $\mathrm{P}(k+1)$ is true when $\mathrm{P}(k)$ is true.

Hence, by principle of mathematical induction, $\mathrm{P}(n)$ is true for every positive integer n.

Example 3:

For all $n\geq 1$, prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Solution

Write,

$$\mathrm{P}(n):\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Note that $\mathrm{P}(1):\frac{1}{1.2}=\frac{1}{2}=\frac{1}{1+1}$, which is true.

Thus, $\mathrm{P}(n)$ is true for $n=1$.

Assume that $\mathrm{P}(k)$ is true for some natural number k,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

Using equation (1),

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{\left(k^2+2k+1\right)}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$$

Therefore, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Hence, by the principle of mathematical induction, $\mathrm{P}(n)$ is true for all natural numbers.

Example 4:

For every positive integer n, prove that 7^n-3^n is divisible by 4

Solution

To write,

$\mathrm{P}(n): 7^n-3^n$ is divisible by 4

Note that,

$\mathrm{P}(1): 7^1-3^1=4$ which is divisible by 4.

Thus $\mathrm{P}(n)$ is true for $n=1$

$\mathrm{P}(k)$ be true for some natural number k,

$\mathrm{P}(k): 7^k-3^k$ is divisible by 4

To write $7^k-3^k=4d$, where $d\in\mathbf{N}$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Now

$$\begin{aligned}
7^{(k+1)}-3^{(k+1)} &= 7^{(k+1)}-7.3^k+7.3^k-3^{(k+1)} \\
&= 7\left(7^k-3^k\right)+(7-3)3^k \\
&= 7(4d)+(7-3)3^k \\
&= 7(4d)+4.3^k \\
&= 4\left(7d+3^k\right)
\end{aligned}$$

From the last line,

$7^{(k+1)}-3^{(k+1)}$ is divisible by 4.

Hence, $\mathrm{P}(k+1)$ is true when $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

The statement is true for each positive integer n.

Example 5:

Prove that $(1+x)^n \geq (1+nx)$, for all natural number n, where $x > -1$.

Solution

Let $\mathrm{P}(n)$ be the given statement,

$\mathrm{P}(n) : (1+x)^n \geq (1+nx)$, for $x > -1$.

Note that $\mathrm{P}(n)$ is true when $n = 1$,

since $(1+x) \geq (1+x)$ for $x > -1$

Assume that,

$\mathrm{P}(k) : (1+x)^k \geq (1+kx), x > -1$ is true. $\rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true for $x > -1$ whenever $\mathrm{P}(k)$ is true. $\rightarrow (2)$

Consider the identity

$$(1+x)^{k+1} = (1+x)^k (1+x)$$

Given that $x > -1$, so $(1+x) > 0$.

Therefore, by using $(1+x)^k \geq (1+kx)$, we have

$$(1+x)^{k+1} \geq (1+kx)(1+x)$$

$$(1+x)^{k+1} \geq \left(1+x+kx+kx^2\right) \qquad \rightarrow (3)$$

Here k is a natural number and $x^2 \geq 0$ so that $kx^2 \geq 0$.

Therefore,

$$\left(1+x+kx+kx^2\right) \geq (1+x+kx)$$

and so obtain

$$(1+x)^{k+1} \geq (1+x+kx)$$

$$(1+x)^{k+1} \geq [1+(1+k)x],$$

Thus, the statement in (2) is established.

Hence, by the principle of mathematical induction, $\mathrm{P}(n)$ is true for all natural numbers.

Example 6: Prove that

$2.7^n + 3.5^n - 5$ is divisible by 24, for all $n \in \mathbf{N}$.

Solution:

Let the statement $P(n)$ be defined as

$P(n): 2.7^n + 3.5^n - 5$ is divisible by 24.

Note that $P(n)$ is true for $n = 1$, since $2.7 + 3.5 - 5 = 24$, which is divisible by 24 .

Assume that $P(k)$ is true

i.e. $2.7^k + 3.5^k - 5 = 24q$, when $q \in \mathbf{N}$ $\rightarrow (1)$

Prove that $P(k+1)$ is true whenever $P(k)$ is true.

We have

$2.7^{k+1} + 3.5^{k+1} - 5 = 2.7^k \cdot 7^1 + 3 \cdot 5^k \cdot 5^1 - 5$

$= 7\left[2.7^k + 3.5^k - 5 - 3.5^k + 5\right] + 3.5^k \cdot 5 - 5$

$= 7\left[24q - 3.5^k + 5\right] + 15.5^k - 5$

Multiply $\left[24q - 3.5^k + 5\right]$ by 7,

$= 7 \times 24q - 21.5^k + 35 + 15.5^k - 5$

$= 7 \times 24q - 6.5^k + 30$

$= 7 \times 24q - 6\left(5^k - 5\right)$

$\left(5^k - 5\right)$ is a multiple of 4,

$= 7 \times 24q - 6(4p)$

$= 7 \times 24q - 24p$

$= 24(7q - p)$

$= 24 \times r; r = 7q - p$, is some natural number.

The expression on the R.H.S. of (1) is divisible by 24.

Thus $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By principle of mathematical induction,

$P(n)$ is true for all $n \in \mathrm{N}$.

Example 7: Prove that

$$1^2+2^2+\ldots+n^2>\frac{n^3}{3}, n\in \mathbf{N}$$

Solution:

Given that,

$$\mathrm{P}(n): 1^2+2^2+\ldots+n^2>\frac{n^3}{3}, n\in \mathbf{N}$$

Note that $\mathrm{P}(n)$ is true for $n=1$ since $1^2>\frac{1^3}{3}$

Assume that $\mathrm{P}(k)$ is true

$$\mathrm{P}(k): 1^2+2^2+\ldots+k^2>\frac{k^3}{3} \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

We have $1^2+2^2+3^2+\ldots+k^2+(k+1)^2$

$$=\left(1^2+2^2+\ldots+k^2\right)+(k+1)^2>\frac{k^3}{3}+(k+1)^2$$

$$=\frac{1}{3}\left[k^3+3k^2+6k+3\right]$$

$$=\frac{1}{3}\left[(k+1)^3+3k+2\right]>\frac{1}{3}(k+1)^3$$

Therefore,

$\mathrm{P}(k+1)$ is also true whenever $\mathrm{P}(k)$ is true.

Hence, by mathematical induction,

$\mathrm{P}(n)$ is true for all $n\in \mathbf{N}$.

Example 8:

Prove the rule of exponents $(ab)^n=a^nb^n$ by using principle of mathematical induction for every natural number.

Solution:

Let $\mathrm{P}(n)$ be the given statement,

$$\mathrm{P}(n): (ab)^n=a^nb^n.$$

Note that $\mathrm{P}(n)$ is true for $n=1$ since $(ab)^1=a^1b^1$.

Let $\mathrm{P}(k)$ be true,

$$(ab)^k = a^k b^k$$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

We have

$$(ab)^{k+1} = (ab)^k (ab)$$

$$= \left(a^k b^k\right)(ab)$$

$$= \left(a^k \cdot a^1\right)\left(b^k \cdot b^1\right)$$

$$= a^{k+1} \cdot b^{k+1}$$

Therefore,

$\mathrm{P}(k+1)$ is also true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By principle of mathematical induction, $\mathrm{P}(n)$ is true for all $n \in \mathrm{N}$.

Exercise 4.1

Question 1:

Prove the following by using the principle of mathematical induction for all $n \in N$

$$1+3+3^2+\ldots+3^{n-1} = \frac{\left(3^n-1\right)}{2}$$

Solution:

The given statement be $\mathrm{P}(n)$,

$$P(n) = 1+3+3^2+\ldots+3^{n-1} = \frac{\left(3^n-1\right)}{2}$$

For $n=1$, we have

$P(1) = \dfrac{\left(3^1-1\right)}{2} = \dfrac{3-1}{2} = \dfrac{2}{2} = 1$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1+3+3^2+\ldots+3^{i-1} = \frac{\left(3^i-1\right)}{2} \quad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$1+3+3^2+\ldots+3^{k-1}+3^{(k+1)-1}$$
$$=\left(1+3+3^2+\ldots+3^{k-1}\right)+3^k$$

Using equation (1),

$$=\frac{\left(3^x-1\right)}{2}+3^i$$

$$=\frac{\left(3^x-1\right)+2.3^x}{2}$$

$$=\frac{(1+2)3^x-1}{2}$$

$$=\frac{3.3^x-1}{2}$$

$$=\frac{3^{2+1}-1}{2}$$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Therefore, by the principle of mathematical induction, statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 2:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^1+2^1+3^1+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Solution:

Given statement be $\mathrm{P}(n)$,

$$\mathrm{P}(n)=1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

For $n=1$, we have

$P(1): 1^3=1=\left(\frac{1(1+1)}{2}\right)^2=\left(\frac{1.2}{2}\right)^2=1^2=1$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1^3+2^3+3^3+\ldots+k^3=\left(\frac{k(k+1)}{2}\right)^2 \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true. Consider

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3$$

$$=\left(1^3+2^3+3^3+\ldots+k^3\right)+(k+1)^3$$

Using equation (1),

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2\left\{k^2+4(k+1)\right\}}{4}$$

Simplify,

$$=\frac{(k+1)^2\left\{k^2+4k+4\right\}}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2(k+1+1)^2}{4}$$

$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 3:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots n)}=\frac{2n}{(n+1)}$$

Solution:

Let the given statement be $P(n)$,

$$\mathrm{P}(n):1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots n}=\frac{2n}{n+1}$$

For $n=1$, we have

$P(1):1=\frac{2.1}{1+1}=\frac{2}{2}=1$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1+\frac{1}{1+2}+\ldots+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1}\rightarrow(1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots k}\right)+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

Using equation (1),

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{\frac{(k+1)(k+1+1)}{2}}$$

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left[\frac{(k+1)^2}{k+2}\right]$$

$= \frac{2(k+1)}{(k+2)}$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $P(n)$ is true for all natural numbers N.

Question 4:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$1.2.3+2.3.4+\ldots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$

Solution:

Given statement be $P(n)$, i.e.,

$$P(n): 1.2.3+2.3.4+\ldots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

For $n=1$, we have $P(1): 1.2.3=6=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1.2.3.4}{4}=6$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1.2.3+2.3.4+\ldots+k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4} \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider.

$1.2.3+2.3.4+\ldots+k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$=\{1.2.3+2.3.4+\ldots+k(k+1)(k+2)\}+(k+1)(k+2)(k+3)$

Using equation (1),

$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ **By the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.**

Question 5:

Prove the following by using the principle of mathematical induction for al $n \in N$:

$$1.3 + 2.3^2 + 3.3^1 + \ldots + n.3^n = \frac{(2n-1)3^{y-1} + 3}{4}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$\mathrm{P}(n):\quad 1.3 + 2.3^2 + 3.3^3 + \ldots + n3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

For $n = 1$, we have $P(1): 1.3 = 3 = \frac{(2.1-1)3^{3+1} + 3}{4} = \frac{3^2 + 3}{4} = \frac{12}{4} = 3$, which is true.

Let $\mathrm{P}(k)$ be true for some positive integer k,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + k3^k = \frac{(2k-1)3^{k+1} + 3}{4} \qquad \to (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider

$$1.3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \ldots + k \cdot 3^k + (k+1) \cdot 3^{k+1} = \left(1.3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \ldots + k \cdot 3^k\right) + (k+1) \cdot 3^{k+1}$$

Using equation (1),

$$= \frac{(2k-1)3^{k+1} + 3}{4} + (k+1)3^{k+1}$$

$$= \frac{(2k-1)3^{k+1} + 3 + 4(k+1)3^{k+1}}{4}$$

$$= \frac{3^{k+1}\{2k - 1 + 4(k+1)\} + 3}{4}$$

$$= \frac{3^{k+1}\{6k + 3\} + 3}{4}$$

$$= \frac{3^{k+1} \cdot 3\{2k + 1\} + 3}{4}$$

$$= \frac{3^{(k+1)+1}\{2k + 1\} + 3}{4}$$

$=\dfrac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

statement $P(n)$ is true for all natural numbers i.e., N.

Question 6:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Solution:

Given statement be $P(n)$,

$$\mathrm{P}(n): \quad 1.2+2.3+3.4+\ldots+n\cdot(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

For $n=1$, we have

$P(1): \quad 1.2=2=\dfrac{1(1+1)(1+2)}{3}=\dfrac{1.2.3}{3}=2$, which is true.

Let $\mathrm{P}(k)$ be true for some positive integer k,

$$1.2+2.3+3.4+\ldots..+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \quad \rightarrow(1)$$

Prove that $\mathrm{P}(k+1)$ is true,

Consider

$1.2+2.3+3\cdot4+\ldots+k\cdot(k+1)+(k+1)\cdot(k+2)=[1.2+2\cdot3+3\cdot4+\ldots+k\cdot(k+1)]+(k+1)\cdot(k+2)$

Using equation (1),

$=\dfrac{k(k+1)(k+2)}{3}+(k+1)(k+2)$

$=(k+1)(k+2)\left(\dfrac{k}{3}+1\right)$

$=\dfrac{(k+1)(k+2)(k+3)}{3}$

$=\dfrac{(k+1)(k+1+1)(k+1+2)}{3}$

Hence, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 7:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\frac{n\left(4n^2+6n-1\right)}{3}$$

Solution:

Given statement be $P(n)$, i.e.,

$$\mathrm{P}(n): \quad 1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\frac{n\left(4n^2+6n-1\right)}{3}$$

For $n=1$,

$$\mathrm{P}(1): 1.3=3=\frac{1\left(4.1^2+6.1-1\right)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3, \text{ which is true.}$$

Let $\mathrm{P}(k)$ be true for some positive integer k, i.e.,

$$1.3+3.5+5.7+\ldots\ldots+(2k-1)(2k+1)=\frac{k\left(4k^2+6k-1\right)}{3} \quad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider

$$(1.3+3.5+5.7+\ldots+(2k-1)(2k+1)+\{2(k+1)-1\}\{2(k+1)+1)$$

Using equation (1),

$$=\frac{k\left(4k^2+6k-1\right)}{3}+(2k+2-1)(2k+2+1)$$

$$=\frac{k\left(4k^2+6k-1\right)}{3}+(2k+1)(2k+3)$$

$$=\frac{k\left(4k^2+6k-1\right)}{3}+\left(4k^2+8k+3\right)$$

$$=\frac{k\left(4k^2+6k-1\right)+3\left(4k^2+8k+3\right)}{3}$$

Multiply $\left(4k^2+6k-1\right)$ by k ,and $\left(4k^2+8k+3\right)$ by 3,

$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

Take common terms,

$$=\frac{k\left(4k^2+14k+9\right)+1\left(4k^2+14k+9\right)}{3}$$

$$=\frac{(k+1)\left(4k^2+14k+9\right)}{3}$$

$$=\frac{(k+1)\left\{4k^2+8k+4+6k+6-1\right\}}{3}$$

$$=\frac{(k+1)\left\{4\left(k^2+2k+1\right)+6(k+1)-1\right\}}{3}$$

$$=\frac{(k+1)\left\{4(k+1)^2+6(k+1)-1\right\}}{3}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $P(n)$ is true for all natural numbers N.

Question 8:

Prove the following by using the principle of mathematical induction for all $n \in N: 1.2+$

$2.2^2+3\cdot 2^2+\ldots+n\cdot 2^n=(n-1)2^{n+1}+2$

Solution:

Let the given statement be $P(n)$,

$\mathrm{P}(n): 1.2+2.2^2+3.2^2+\ldots+n\cdot 2^n=(n-1)2^{n+1}+2$

For $n=1$, we have

$P(1): 1.2=2=(1-1)2^{1+1}+2=0+2=2$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$1.2+2.2^2+3.2^2+\ldots+k\cdot 2^k=(k-1)2^{k+1}+2\ldots$ (i)

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$\left\{1\cdot 2+2\cdot 2^2+3\cdot 2^3+\ldots+k\cdot 2^k\right\}+(k+1)\cdot 2^{k+1}$$

$$=(k-1)2^{k+1}+2+(k+1)2^{k+1}$$

$$=2^{k+1}\{(k-1)+(k+1)\}+2$$

$$=2^{k+1}\cdot 2k+2$$

$$=k\cdot 2^{(k+1)-1}+2$$

$$=\{(k+1)-1\}2^{(k+1)-1}+2$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $P(n)$ is true for all natural numbers N.

Question 9:

Prove the following by using the principle of mathematical induction for all $n\in N$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Solution:

Given statement be $P(n)$, i.e.,

$$\mathrm{P}(n)=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

For $n=1$, we have

$P(1):\ \frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$, which is true.

Let $\mathrm{P}(k)$ be true for some positive integer k, i.e.,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$

$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$

$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$

$=1-\frac{1}{2^{k+1}}$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $P(n)$ is true, for all natural numbers N.

Question 10:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Solution:

Given statement be $P(n)$,

$$\mathrm{P}(n)=\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

For $n=1$, we have

$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

Using equation (1),

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{6k+10}\right)$$

$$=\frac{(k+1)}{6k+10}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 11:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Solution:

Given statement be $\mathrm{P}(n)$,

$$\mathrm{P}(n)=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

For $n=1$, we have $\mathrm{P}(1):\frac{1}{1\cdot 2\cdot 3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot 4}{4\cdot 2\cdot 3}=\frac{1}{1\cdot 2\cdot 3}$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$$

Prove that $\mathrm{P}(k+1)$ is true,

Consider,

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots.+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+6k+9\right)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Hence, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction, statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 12:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r-1}$$

Solution:

Given statement be $\mathrm{P}(n)$,

$$\mathrm{P}(n): a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r-1}$$

For $n = 1$, we have

$\mathrm{P}(1): a = \dfrac{a\left(r' - 1\right)}{(r-1)} = a$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$a + ar + ar^2 + \ldots\ldots + ar^{k-1} = \frac{a\left(r^k - 1\right)}{r-1} \qquad \to (1)$$

$$\left\{a + ar + ar^2 + \ldots\ldots + ar^{k-1}\right\} + ar^{(k+1)-1} = \frac{a\left(r^k - 1\right)}{r-1} + ar^k$$

$$= \frac{a\left(r^k - 1\right) + ar^k(r-1)}{r-1}$$

$$= \frac{a\left(r^k - 1\right) + ar^{k+1} - ar^k}{r-1}$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$$

$$= \frac{ar^{k+1} - a}{r-1}$$

$$= \frac{a\left(r^{k+1} - 1\right)}{r-1}$$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 13:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

Solution:

Given statement be $P(n)$,

$$\mathrm{P}(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

For $n=1$, we have,

$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2$$

Prove that $\mathrm{P}(k+1)$ is true,

Consider,

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

Using equation (1),

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

$$=(k+1)^2+2(k+1)+1$$

$$=\{(k+1)+1\}^2$$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 14:

Prove the following by using the principle of mathematical induction for all $n\in N$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

Solution:

Given statement be $\mathrm{P}(n)$,

$$\mathrm{P}(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

For $n=1$, we have

$\mathrm{P}(1):\left(1+\frac{1}{1}\right)=2=(1+1)$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\mathrm{P}(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

Using equation (1),

$$=(k+1)\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

$$=(k+1)+1$$

Hence, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $P(n)$ is true for all natural numbers N.

Question 15:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution:

Given statement be $P(n)$,

$$\mathrm{P}(n)=1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

For $n=1$, we have

$\mathrm{P}(1)=1^2=1=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\mathrm{P}(k)=1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \qquad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true.

Consider,

$$\left\{1^2+3^2+5^2+\ldots+(2k-1)^2\right\}+\{2(k+1)-1\}^2$$

Using equation (1),

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+2-1)^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$$

$$=\frac{(2k+1)\left\{2k^2-k+6k+3\right\}}{3}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 16:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Solution:

Given statement be $P(n)$

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$, we have $P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}$, which is true.

$P(k)$ be true for some positive integer k,

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \qquad \rightarrow (1)$$

Prove that $P(k+1)$ is true,

Consider,

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

Using equation (1),

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $P(n)$ is true for all natural numbers N.

Question 17:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Solution:

Given statement be $\mathrm{P}(n)$,

$$\mathrm{P}(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

For $n=1$, we have $\mathrm{P}(1): \frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$, which is true.

$\mathrm{P}(k)$ be true for some positive integer k,

$$\mathrm{P}(k): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \quad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true,

Consider,

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $P(n)$ is true for all natural numbers N.

Question 18:

Prove the following by using the principle of mathematical induction for a $n \in N$:

$$1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

Solution:

Given statement be $P(n)$,

$$\mathrm{P}(n): 1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

Noted that $\mathrm{P}(n)$ is true for $n=1$ since,

$$1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}$$

$\mathrm{P}(k)$ be true for some positive integer k,

$$1+2+\ldots+k<\frac{1}{8}(2k+1)^2 \quad \rightarrow (1)$$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Consider,

$$(1+2+\ldots+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$$

Using equation (1),

$$<\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$$

$$<\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$$

$$<\frac{1}{8}\left\{4k^2+12k+9\right\}$$

$$<\frac{1}{8}(2k+3)^2$$

$< \frac{1}{8}\{2(k+1)+1\}^2$

Hence, $(1+2+3+\ldots+k)+(k+1) < \frac{1}{8}(2k+1)^2+(k+1)$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 19:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$n(n+1)(n+5)$ is a multiple of 3.

Solution:

Given statement be $P(n)$,

$\mathrm{P}(n): n(n+1)(n+5)$, which is a multiple of 3.

Noted that $P(n)$ is true for $n=1$ since $1(1+1)(1+5)=12$,

Which is a multiple of 3.

$=3\{m+(k+1)(k+4)\}=3\times q$, where $q=\{m+(k+1)(k+4)\}$ is some natural number

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Let $\mathrm{P}(k)$ be true for some positive integer k,

$k(k+1)(k+5)$ is a multiple of 3.

$\therefore k(k+1)(k+5)=3m$, where $m \in \mathbf{N}$ $\rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$(k+1)\{(k+1)+1\}\{(k+1)+5\}$

$=(k+1)(k+2)\{(k+5)+1\}$

$=(k+1)(k+2)(k+5)+(k+1)(k+2)$

$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$

$=3m+(k+1)\{2(k+5)+(k+2)\}$

$=3m+(k+1)\{2k+10+k+2\}$

$=3m+(k+1)(3k+12)$

$=3m+3(k+1)(k+4)$

$=3\{m+(k+1)(k+4)\}=3\times q$, where $q=\{m+(k+1)(k+4)\}$ is some natural number

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3

Hence, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N .

Question 20:

Prove the following by using the principle of mathematical induction for all $n\in N$:

$10^{2n-1}+1$ is divisible by 11

Solution:

Given statement be $P(n)$,

$P(n):10^{2n-1}+1$ is divisible by 11

It can be observed that $\mathrm{P}(n)$ is true for $n=1$

Since $P(1)=10^{2.1-1}+1=11$, which is divisible by 11,

Let $\mathrm{P}(k)$ be true for some positive integer k ,

$10^{2k-1}+1$ is divisible by 11

$\therefore 10^{2k-1}+1=11m$, where $m\in\mathbf{N} \qquad \rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Consider,

$10^{2(k+1)-1}+1$

$=10^{2k+2-1}+1$

$=10^{2k+1}+1$

$=10^2\left(10^{2k-1}+1-1\right)+1$

$=10^2\left(10^{2k-1}+1\right)-10^2+1$

Using equation (1) ,

$=10^2\cdot 11m-100+1$

$=100\times 11m-99$

$=11(100m-9)$

$=11r$, where $r=(100m-9)$ is some natural number

Therefore, $10^{2(\xi+1)-1}+1$ is divisible by 11.

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 21:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$x^{2n}-y^{2n}$ is divisible by $x+y$

Solution:

Given statement be $P(n)$,

$P(n): x^{2n}-y^{2n}$ is divisible by $x+y$.

It can be observed that $\mathrm{P}(n)$ is true for $n=1$.

This is so because $x^{2\times1}-y^{2\times1}=x^2-y^2=(x+y)(x-y)$ is divisible by $(x+y)$

Let $\mathrm{P}(k)$ be true for some positive integer k, i.e.,

$x^{2k}-y^{2k}$ is divisible by $x+y$.

$\therefore$ Let $x^{2k}-y^{2k}=m(x+y)$, where $m \in \mathbf{N}$ $\quad \rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Consider,

$x^{2(k+1)}-y^{2(k+1)}$

$=x^{2k}\cdot x^2-y^{2k}\cdot y^2$

$=x^2\left(x^{2k}-y^{2k}+y^{2k}\right)-y^{2k}\cdot y^2$

$=x^2\left\{m(x+y)+y^{2k}\right\}-y^{2k}\cdot y^2$

Using equation (1),

$=m(x+y)x^2+y^{2k}\cdot x^2-y^{2k}\cdot y^2$

$=m(x+y)x^2+y^{2k}\left(x^2-y^2\right)$

$=m(x+y)x^2+y^{2k}(x+y)(x-y)$

$= (x+y)\left\{mx^2 + y^{2k}(x-y)\right\}$, which is a factor of $(x+y)$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 22:

Prove the following by using the principle of mathematical induction for all $n \in N: 3^{2n+2} - 8n - 9$ is divisible by 8.

Solution:

Given statement be $P(n)$,

$P(n): 3^{2n+2} - 8n - 9$ is divisible by 8.

$\mathrm{P}(n)$ is true for $n = 1$

since $3^{2\times1+2} - 8\times1 - 9 = 64$, which is divisible by 8.

Let $P(k)$ be true for some positive integer k,

$3^{2k+2} - 8k - 9$ is divisible by 8,

$\therefore 3^{2k+2} - 8k - 9 = 8m;$ Where $m \in \mathbf{N}$ $\quad\quad \rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true,

Consider,

$3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 3^2\left(3^{2k+2} - 8k - 9 + 8k + 9\right) - 8k - 17$

$= 3^2\left(3^{2k+2} - 8k - 9\right) + 3^2(8k+9) - 8k - 17$

$= 9.8m + 9(8k+9) - 8k - 17$

$= 9.8m + 72k + 81 - 8k - 17$

$= 9.8m + 64k + 64$

$= 8(9m + 8k + 8)$

$= 8r$, where $r = (9m + 8k + 8)$ is a natural number

Therefore,

$3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N.

Question 23:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$41^n - 14^n$ is a multiple of 27

Solution:

Given statement be $\mathrm{P}(n)$,

$\mathrm{P}(n): 41^n - 14^n$ is a multiple of 27

$\mathrm{P}(n)$ is true for $n=1$

Since $41^1 - 14^1 = 27$, which is a multiple of 27.

Let $\mathrm{P}(k)$ be true for some positive integer k,

$41^k - 14^k$ is a multiple of 27

$\therefore 41^k - 14^k = 27m$, where $m \in \mathbf{N}$ $\rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true,

$41^{k+1} - 14^{k+1}$

$= 41^k \cdot 41 - 14^k \cdot 14$

$= 41\left(41^k - 14^k + 14^k\right) - 14^k \cdot 14$

$= 41\left(41^k - 14^k\right) + 41.14^k - 14^k \cdot 14$

$= 41.27m + 14^k(41-14)$

$= 41.27m + 27.14^k$

$= 27\left(41m - 14^k\right)$

$= 27 \times r$, where $r = \left(41\mathrm{m} - 14^k\right)$ is a natural number Therefore, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

$\therefore$ By the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers, N.

Question 24:

Prove the following by using the principle of mathematical induction for all $n \in \mathrm{N}$:

$(2n+7) < (n+3)^2$

Solution:

Given statement be $P(n)$,

$\mathrm{P}(n):(2n+7) < (n+3)^2$

It can be observed that $\mathrm{P}(n)$ is true for $n=1$

Since $2.1+7=9<(1+3)^2=16$, which is true.

Let $\mathrm{P}(k)$ be true for some positive integer k,

$(2k+7) < (k+3)^2 \quad \rightarrow (1)$

Prove that $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Consider,

$\{2(k+1)+7\} = (2k+7)+2$

Using equation (1),

$\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2+2$

$2(k+1)+7 < k^2+6k+9+2$

$2(k+1)+7 < k^2+6k+11$

Now,

$k^2+6k+11 < k^2+8k+16$

Therefore, $2(k+1)+7 < (k+4)^2$

$2(k+1)+7 < \{(k+1)+3\}^2$

Thus, $\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true.

Hence, by the principle of mathematical induction,

Statement $\mathrm{P}(n)$ is true for all natural numbers N .