Chapter 8: Binomial Theorem

Exercise 8.1

1:

The given expression is $(1-2x)^5$

We need to find the binomial expansion of the given expression

The general expansion using binomial theorem is found by $(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$

Substituting $a=1 \,\&\, b=-2x; n=5$ in the equation

$$(1+(-2x))^5 = \sum_{k=0}^{5} {}^5C_k . 1^{5-k}(-2x)^k$$

$$(1-2x)^5$$
$$=\left({}^5C_0(1)^{5-0}(-2x)^0\right)+\left({}^5C_1(1)^{5-1}(-2x)^1\right)+\left({}^5C_2(1)^{5-2}(-2x)^2\right)+\left({}^5C_3(1)^{5-3}(-2x)^3\right)+\left({}^5C_4(1)^{5-4}(-2x)^4\right)+\left({}^5C_5(1)^{5-5}(-2x)^5\right)$$
$$=(1.1.1)+(5(1)(-2x))+\left(\left(\frac{5\text{x}4}{2}\right).1.4x^2\right)+\left(\left(\frac{5\text{x}4}{2}\right).1.(-8x^3)\right)+\left(5(1)(16x^4)\right)+\left(1.(1)(-32x^5)\right) \qquad \because {}^nC_r = \frac{n!}{(n-r)!r!}$$
$$=1-10x+\left(10.4x^2\right)+\left(10.(-8x^3)\right)+\left(80x^4-32x^5\right)$$
$$=1-10x+40x^2-80x^3+80x^4-32x^5$$
is the expansion of $(1-2x)^5$.

2:

The given expression is $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

We need to find the binomial expansion of the given expression

The general expansion using binomial theorem is found by $(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$

Substituting $a=\frac{2}{x} \,\&\, b=\frac{-x}{2}; n=5$ in the equation

$$\left(\frac{2}{x}+\left(\frac{-x}{2}\right)\right)^5=\sum_{k=0}^{5}{}^5C_k\left(\frac{2}{x}\right)^{5-k}\left(\frac{-x}{2}\right)^k$$

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5$$

$$=\left({}^5C_0\left(\frac{2}{x}\right)^{5-0}\left(\frac{-x}{2}\right)^0\right)+\left({}^5C_1\left(\frac{2}{x}\right)^{5-1}\left(\frac{-x}{2}\right)^1\right)+\left({}^5C_2\left(\frac{2}{x}\right)^{5-2}\left(\frac{-x}{2}\right)^2\right)+\left({}^5C_3\left(\frac{2}{x}\right)^{5-3}\left(\frac{-x}{2}\right)^3\right)+\left({}^5C_4\left(\frac{2}{x}\right)^{5-4}\left(\frac{-x}{2}\right)^4$$

$$=\left(1.\frac{32}{x^5}.1\right)+\left(5.\frac{16}{x^4}.\left(\frac{-x}{2}\right)\right)+\left(\left(\frac{5x4}{2}\right).\frac{8}{x^3}.\left(\frac{x^2}{4}\right)\right)+\left(\left(\frac{5x4}{2}\right).\frac{4}{x^2}.\left(\frac{-x^3}{8}\right)\right)+\left(5.\frac{2}{x}.\frac{x^2}{4}\right)+\left(1.(1)\left(\frac{-x}{2}\right)\right) \qquad \because {}^nC_r$$

$$=\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}$$

is the expansion of $\left(\frac{2}{x}-\frac{x}{2}\right)^5$.

3.

The given expression is $(2x-3)^6$

We need to find the binomial expansion of the given expression

The general expansion using binomial theorem is found by $(a+b)^n=\sum_{k=0}^{n}{}^nC_k a^{n-k}b^k=\sum_{k=0}^{n}{}^nC_k a^k b^{n-k}$

Substituting $a=2x\ \&\ b=-3; n=6$ in the equation

$$(2x-3)^6=\sum_{k=0}^{6}{}^6C_k(2x)^{6-k}(-3)^k$$

$$(2x+(-3))^6$$

$$=\left({}^6C_0(2x)^6(-3)^0\right)+\left({}^6C_1(2x)^{6-1}(-3)^1\right)+\left({}^6C_2(2x)^{6-2}(-3)^2\right)+\left({}^6C_3(2x)^{6-3}(-3)^3\right)+\left({}^6C_4(2x)^{6-4}(-3)^4\right)+\left({}^6C_5(2x)^{6-5}(-3)^5\right)+\left({}^6C_6(2x)^{6-6}(-3)^6\right)$$

$$=\left(1.(64x^6).1\right)+\left((32x^5)(-18)\right)+\left(\left(\frac{6x5}{2}\right)(16x^4).9\right)+\left(\left(\frac{6x5x4}{3}\right)(8x^3)(-27)\right)+\left(\left(\frac{6x5}{2}\right)(4x^2)(81)\right)+\left(6.(2x)(-243)\right)+729 \qquad \because {}^nC_r=\frac{n!}{(n-r)!r!}$$

$$=64x^6-576x^5+2160x^4+8640x^3+4860x^2-2916x+729$$

is the expansion of $(2x-3)^6$.

4.

The given expression is $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

We need to find the binomial expansion of the given expression

The general expansion using binomial theorem is found by $(a+b)^n=\sum_{k=0}^{n}{}^nC_k a^{n-k}b^k=\sum_{k=0}^{n}{}^nC_k a^k b^{n-k}$

Substituting $a=\frac{x}{3}$ & $b=\frac{1}{x}; n=5$ in the equation

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5=\sum_{k=0}^{5}{}^5C_k\left(\frac{x}{3}\right)^{5-k}\left(\frac{1}{x}\right)^k$$

$$=\left({}^5C_0\left(\frac{x}{3}\right)^{5-0}\left(\frac{1}{x}\right)^0\right)+\left({}^5C_1\left(\frac{x}{3}\right)^{5-1}\left(\frac{1}{x}\right)^1\right)+\left({}^5C_2\left(\frac{x}{3}\right)^{5-2}\left(\frac{1}{x}\right)^2\right)+\left({}^5C_3\left(\frac{x}{3}\right)^{5-3}\left(\frac{1}{x}\right)^3\right)+\left({}^5C_4\left(\frac{x}{3}\right)^{5-4}\left(\frac{1}{x}\right)^4\right)+\left({}^5C_5\left(\frac{x}{3}\right)^{5-5}\left(\frac{1}{x}\right)^5\right)$$

$$=\left(1.\frac{x^5}{243}.1\right)+\left(5.\frac{x^4}{81}.\left(\frac{1}{x}\right)\right)+\left(\left(\frac{5x4}{2}\right).\frac{x^3}{27}\left(\frac{1}{x^2}\right)\right)+\left(\left(\frac{5x4}{2}\right).\frac{x^2}{9}.\left(\frac{1}{x^3}\right)\right)+\left(5.\frac{x}{3}\left(\frac{1}{x^4}\right)\right)+\left(1.\frac{x}{3}\left(\frac{1}{x^5}\right)\right) \qquad \because {}^nC_r=\frac{n!}{(n-r)!r!}$$

$$=\frac{x^5}{243}+\frac{5x^3}{81}+\frac{10x}{27}+\frac{10}{9x}+\frac{5}{3x^3}+\frac{1}{x^5}$$

is the expansion of $\left(\frac{x}{3}+\frac{1}{x}\right)^5$.

Exercise 8.2

1. Find the coefficient of x^5 **in** $(x+3)^8$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1}={}^nC_ra^{n-r}b^r$$

Assuming that x^5 occurs in the $(r+1)^{th}$ term of the expansion $(x+3)^8$, we obtain

$$T_{r+1}={}^8C_r(x)^{8-r}(3)^r$$

Comparing the indices of x in x^5 and in T_{r+1},

we obtain $r=3$

Thus, the coefficient of x^5 is

$${}^nC_3(3)^3=\frac{8!}{3!5!}\times3^3$$

$$=\frac{8\cdot7\cdot6\cdot5!}{3\cdot2\cdot5!}\cdot3^3$$

$$=1512$$

2. Find the coefficient of a^5b^7 in $(a-2b)^{12}$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Assuming that a^5b^7 occurs in the $(r+1)^{th}$ term of the expansion $(a-2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r}(-2b)^r$$

$$= {}^{12}C_r (-2)^r (a)^{12-r}(b)^r$$

Comparing the indices of a and b in a^5b^7 and in T_{r+1},

we obtain $r = 7$

Thus, the coefficient of a^5b^7 is

$${}^{12}C_7(-2)^7 = \frac{12!}{7!5!} \times 2^7$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 7!} \cdot 2^7$$

$$= -(792)(128)$$

$$= -101376$$

3. Write the general term in the expansion of $(x^2 - y)^6$

Solution:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6C_r (x^2)^{6-r}(-y)^r$$

$$T_{r+1} = (-1)^r\, {}^6C_r . x^{12-2r} . y^r$$

4. Write the general term in the expansion of $(x^2 - yx)^{12}, x \neq 0$

Solution:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Thus, the general term in the expansion of $(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r (x^2)^{12-r} (-yx)^r$$

$$T_{r+1} = (-1)^r \, {}^{12}C_r . x^{24-2r} . y^r . x^r$$

$$T_{r+1} = (-1)^r \, {}^{12}C_r . x^{24-r} . y^r$$

5. Find the 4^{th} term in the expansion of $(x-2y)^{12}$.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Thus, the 4^{th} term in the expansion of $(x-2y)^{12}$ is

$$T_4 = T_{3+1} = {}^nC_3 (x)^{12-3} (-2y)^3$$

$$= (-1).\left(\frac{12!}{3!9!}\right).x^9.(2)^3.y^3$$

$$= -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3$$

$$= -1760 x^9 y^3$$

6. Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Thus, the 13^{th} terms in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13} = T_{12+1} = {}^{18}C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= (-1)^{12}.\left(\frac{18!}{12!6!}\right)(9)^6 (x)^6 \left(\frac{1}{3}\right)^{12} \left(\frac{1}{\sqrt{x}}\right)^{12}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12! \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \cdot x^6 \cdot \left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right)$$

$=18564$

7. Find the middle terms in the expansions of $\left(3-\frac{x^3}{6}\right)^7$

Solution:

It is known that in the expansion of $(a+b)^n$, if n is odd, then there are two middle terms, namely $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$.

Therefore, the middle terms in the expansion of the expansion $\left(3-\frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th}=4^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th}=5^{th}$ term

$$T_4=T_{3+1}={}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3$$

$$=(-1)^3\frac{7!}{3!4!}.3^4.\frac{x^9}{6^3}$$

$$=-\frac{7\cdot6\cdot5\cdot4!}{4!\cdot3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^9$$

$$=-\frac{105}{8}x^9$$

$$T_5=T_{4+1}={}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3$$

$$=(-1)^4\frac{7!}{3!4!}.3^3.\frac{x^{12}}{6^4}$$

$$=\frac{7\cdot6\cdot5\cdot4!}{4!\cdot3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12}$$

$$=\frac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

8. Find the middle terms in the expansions of $\left(\frac{x}{3}+9y\right)^{10}$.

Solution:

It is known that in the expansion of $(a+b)^n$, if n is even, then there are two middle terms $\left(\frac{n+1}{2}\right)^{th}$ term.

Therefore, the middle terms in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th}=6^{th}$

$$T_6=T_{5+1}={}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5$$

$$=\frac{10!}{5!5!}\cdot\frac{x^5}{3^5}.9^5.y^5$$

$$=\frac{10\cdot9\cdot8\cdot7\cdot6\cdot5!}{5\cdot4\cdot3\cdot2\cdot5!}\cdot\frac{1}{3^5}\cdot3^{10}\cdot x^5\cdot y^5$$

$$=252\times3^5\cdot x^5\cdot y^5$$

$$=61236x^5y^5$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $61236\,x^5y^5$.

9. In the expansion of $(1+a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1}={}^nC_r a^{n-r}b^r$$

.

Assuming that a^m occurs in the $(r+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{r+1}={}^{m+n}C_r(1)^{m+n-r}(a)^r$$

$$={}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in T_{r+1}, we obtain

$$r=m$$

Therefore, the coefficient of a^m is

$$^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \quad \ldots\ldots (1)$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1}, we obtain

$$k = n$$

Therefore, the coefficient of a^n is

$$^{m+n}C_m = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \quad \ldots\ldots (2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

10. The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio $1:3:5$. Find n and r.

Solution:

It is known that $(k+1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{k+1} = {}^nC_k a^{n-k} b^k.$$

Therefore, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is

$$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)}$$

$$T_{r-1} = {}^nC_{r-2} (x)^{n-r+2}$$

$(r+1)$ term in the expansion of $(x+1)^n$ is

$$T_{r+1} = {}^nC_r (x)^{n-r} (1)^r$$

$$T_{r+1} = {}^nC_r (x)^{n-r}$$

r^{th} term in the expansion of $(x+1)^n$ is

$$T_r = {}^nC_{r-1} (x)^{n-(r-1)} (1)^{(r-1)}$$

$$T_r = {}^nC_{r-1} (x)^{n-r+1}$$

Therefore, the coefficients of the $(r-1)^{th}$, r^{th}, and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ are respectively. Since these coefficients are in the ratio $1:3:5$, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!}$$

$$\frac{^nC_{r-2}}{^nC_{r-1}}=\frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!}$$

$$\frac{^nC_{r-2}}{^nC_{r-1}}=\frac{r-1}{n-r+2}$$

$$\therefore\frac{r-1}{n-r+2}=\frac{1}{3}$$

$$\Rightarrow 3r-3=n-r+2$$

$$\Rightarrow n-4r+5=0\ \ldots(1)$$

$$\frac{^nC_{r-1}}{^nC_{r}}=\frac{n!}{(r-1)!(n-r+1)!}\times\frac{r!(n-r)!}{n!}$$

$$\frac{^nC_{r-2}}{^nC_{r-1}}=\frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$\frac{^nC_{r-2}}{^nC_{r-1}}=\frac{r}{n-r+1}$$

$$\therefore\frac{r}{n-r+1}=\frac{3}{5}$$

$$\Rightarrow 5r=3n-3r+3$$

$$\Rightarrow 3n-8r+3=0\ \ldots(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r - 12 = 0$

$\Rightarrow r = 3$

Putting the value of r in (1), we obtain $n - 12 + 5 = 0$

$\Rightarrow n = 7$

Thus, $n = 7$ and $r = 3$

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Solution:

It is known that $(k+1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1}={}^nC_r a^{n-r}b^r$.

Assuming that x^n occurs in the $(r+1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$

Comparing the indices of x in x^n and in T_{r+1}, we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$${}^{2n}C_n = \frac{(2n)!}{n!(2n-n)!}$$

$${}^{2n}C_n = \frac{(2n)!}{n!n!}$$

$${}^{2n}C_n = \frac{(2n)!}{(n!)^2} \quad \text{.... (1)}$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$

Comparing the indices of x in x^n and T_{k+1}, we obtain $k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$${}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!}$$

$${}^{2n-1}C_n = \frac{(2n-1)!}{n!(n-1)!}$$

$${}^{2n-1}C_n = \frac{2n(2n-1)!}{2n.n!(n-1)!}$$

$${}^{2n-1}C_n = \frac{2n!}{2.n!.n!}$$

$${}^{2n-1}C_n = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \quad \text{..... (2)}$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right) = {}^{2n-1}C_n$$

$$\Rightarrow {}^{2n}C_n = 2({}^{2n-1}C_n)$$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

12. Find a positive value of $\mathbf{m}$ for which the coefficient of $\mathbf{x^2}$ in the expansion $(\mathbf{1} + \mathbf{x})^{\mathbf{m}}$ is 6.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion $(1+x)^m$, we obtain

$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain $r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$

It is given that the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

$\therefore {}^mC_2 = 6$

$\Rightarrow \dfrac{m!}{2!(m-2)!} = 6$

$\Rightarrow \dfrac{m(m-1)(m-2)!}{2\times(m-2)!} = 6$

$\Rightarrow m(m-1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m(m-4) + 3(m-4) = 0$

$\Rightarrow (m-4)(m+3) = 0$

$\Rightarrow (m-4) = 0$ or $(m+3) = 0$

$\Rightarrow m = 4$ or $m = -3$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6, is 4.

1: Find a, b and n **in the expansion of** $(a+b)^n$ **if the first three terms of the expansion are** $729, 7290$, **and** 30375 **respectively.**

Answer: We know that $(r+1)^{th}$ term, (T_{r+1}), from the binomial expansion of $(a+b)^n$ is given as $T_{r+1} = {}^nC_r a^{a-r} b^r$

The first three terms of the expansion are given as $729, 7290$, and 30375 respectively. So, we obtain

$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729$................(1)

$T_2 = {}^nC_1 a^{n-1} b^1 = n\, a^{n-1} b = 7290$......................(2)

$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375$......................(3)

(2) / (1), we get

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \text{......................(4)}$$

(3) / (4), we get

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \quad \text{[Using(4)]}$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \text{...................(5)}$$

From equation (4) & (5), we get

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Put $n = 6$ in equation (1), we get a^6

$= 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From equation (5), we get

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$

2: Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer: We know that $(r+1)^{th}$ term, (T_{r+1}), from the binomial expansion of $(a+b)^n$ is given as

$T_{r+1} = {}^nC_r a^{a-r} b^r$

Let x^2 occur in the $(r+1)^{th}$ term, of the expansion $(3+ax)^9$, we get

$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$

Now, we compare the indices of x in x^2 in T_{r+1}, we get

$r = 2$

So, the coefficient of x^2 is

$${}^9C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!}(3)^7 a^2 = 36(3)^7 a^2$$

Let x^3 have in $(k+1)^{th}$ term from the expansion $(3+ax)^9$, we get

$T_{k+1} = {}^9C_k (3)^{9-k} (ax)^k = {}^9C_k (3)^{9-k} a^k x^k$

Now, we compare the indices of x in x^3 in T_{k+1} we get

$k = 3$

So, the coefficient of x^3 is

$${}^9C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!}(3)^6 a^3 = 84(3)^6 a^3$$

Also, we know the coefficient of x^2 and x^3 are same

$84(3)^6 a^3 = 36(3)^7 a^2$

$\Rightarrow 84a = 36 \times 3$

$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$

$$\Rightarrow a = \frac{9}{7}$$

So, the value of $a = \frac{9}{7}$

3: Find the coefficient of x^5 in the product $(1+2x)^6 (1-x)^7$ using binomial theorem.

Answer: By binomial theorem, the given expression can be expanded as

$(1+2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6$

$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$

$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

$\therefore (1+2x)^6 (1-x)^7$

$= \left(1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\right)\left(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7\right)$

$= 171x^5$

Product in the two bracket is not required.

Those terms which terms are include in x^5, are required

Those term which containing x^5

So, the product is 171 of the given coefficient x^5

4: If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand]

Answer: It is required to prove that $a - b$ is a factor of $a^n - b^n$,

Now, we will prove that $a^n - b^n = k(a-b)$, for some natural number k

Let $a = a - b + b$

$a^n = (a - b + b)^n = [(a-b) + b]^n$

$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_n b^n$

$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$

$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$

$\Rightarrow a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$

$\Rightarrow a^n - b^n = k(a-b)$

$k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$ for some natural number

5: Evaluate $(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$

Answer: By binomial theorem given expression $(a+b)^6 - (a-b)^6$ can solved as

$(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$

$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

$(a-b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$

$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

$\therefore (a+b)^6 - (a-b)^6 = 2\left[6a^5b + 20a^3b^3 + 6ab^5\right]$

$a = \sqrt{3}$ and $b = \sqrt{2}$,

Put $a = \sqrt{3}$ and $b = \sqrt{2}$,

$(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 2\left[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5\right]$

$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$

$= 2 \times 198\sqrt{6}$

$= 396\sqrt{6}$

6: Find the value of $\left(a^2 + \sqrt{a^2-1}\right)^4 + \left(a^2 - \sqrt{a^2-1}\right)^4$

Answer: By Binomial theorem,given expression $(x+y)^4 + (x-y)^4$ can be solved as

$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x-y)^y = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$

Put the value of $x = a^2$ and $y = \sqrt{a^2-1}$

$\left(a^2 + \sqrt{a^2-1}\right)^4 + \left(a^2 - \sqrt{a^2-1}\right)^4 = 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$

$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$

$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1\right]$

$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

7: Find an approximation of $(0.99)^5$ **using the first three terms of its expansion.**

Answer:

$0.99 = 1 - 0.01$

$\therefore (0.99)^5 = (1 - 0.01)^5$

$= {}^3C_0(1)^5 - {}^3C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2$ (approximately)

$= 1 - 5(0.01) + 10(0.01)^2$

$= 1 - 0.05 + 0.001$

$= 1.001 - 0.05$

$= 0.951$

Value of $(0.99)^5$ approx. $= 0.951$

8: Find n if the ratio of thefifth term from the beginning to thefifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{2}}\right)^n$ is $\sqrt{6} : 1$

Answer: we know that $(a + b)^n = {}^nC_b a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n$

5^{th} term from the beginning $= {}^nC_4 a^{n-4}b^4$

5^{th} term from the end $= {}^nC_{n-4}a^4b^{n-4}$

So, from the expansion $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{2}}\right)^n$ 5^{th} term from the beginning $= {}^nC_4(\sqrt[4]{2})^{0-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ & 5^{th} term from the end $= {}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^nC_4(\sqrt[4]{2})^{n-4}\left(\frac{1}{\sqrt[3]{3}}\right)^4 = {}^nC_4\frac{(\sqrt[3]{2})^n}{(\sqrt[4]{2})^4}.\frac{1}{3} = {}^nC_4\frac{(\sqrt[3]{2})^n}{2}.\frac{1}{3} = \frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^n \ldots\ldots\ldots\ldots(1)$$

$${}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[3]{3}}\right)^{n-4} = {}^nC_{n-4}2.\frac{(\sqrt[3]{3})^4}{(\sqrt[4]{3})^n} = {}^nC_{n-4}.2.\frac{3}{(\sqrt[3]{3})^n} = \frac{6n!}{(n-4)!4!}.\frac{1}{(\sqrt[3]{3})^n}\ldots\ldots\ldots(2)$$

We have also given that the ratio of the 5^{th} term from the beginning and the 5^{th} term from the end $= \sqrt{6} : 1$

From equation (1) & (2), we get

$$\frac{n!}{6.4!(n-4)!}(\sqrt[3]{2})^n = \frac{6n!}{(n-4)!4!}.\frac{1}{(\sqrt[2]{3})^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6}.\frac{6}{(\sqrt[4]{3})^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6}.\frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

$$\Rightarrow (\sqrt{6})^n = 36\sqrt{6}$$

$$\Rightarrow 6^{n/4} = 6^{5/2}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n - 4 .\frac{5}{2} = 10$$

So, the value of $n = 10$.

9: Expand using Binomial theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, x \neq 0$

Answer: We will use Binomial theorem in the given expression $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, x \neq 0$

$$\left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1 + \frac{x}{2}\right)^4 - {}^4C_1\left(1 + \frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1 + \frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1 + \frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$= \left(1 + \frac{x}{2}\right)^4 - 4\left(1 + \frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1 + x + \frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1 + \frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} ....(1)$$

Now, we will use again Binomial theorem, then

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4.\frac{x}{2} + 6.\frac{x^2}{4} + 4.\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} ....(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} ....(3)$$

From equation (1), (2) & (3)

$$\left[\left(1+\frac{x}{2}\right) - \frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

10: Find the expansion of $\left(3x^2 - 2ax + 3a^2\right)^3$ using binomial theorem.

Answer: given expression $\left(3x^2 - 2ax + 3a^2\right)^3$ can be expressed using of Binomial theorem

$$\left[\left(3x^2 - 2ax\right) + 3a^2\right]^3$$

$$= {}^3C_0\left(3x^2 - 2ax\right)^3 + {}^3C_1\left(3x^2 - 2ax\right)^2\left(3a^2\right) + {}^3C_2\left(3x^2 - 2ax\right)\left(3a^2\right)^2 + 3C_3\left(3a^2\right)^3$$

$$= \left(3x^2 - 2ax\right)^3 + 3\left(9x^4 - 12ax^3 + 4a^2x^2\right)\left(3a^2\right) + 3\left(3x^2 - 2ax\right)\left(9a^4\right) + 27a^6$$

$$= \left(3x^2 - 2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$$

$$= \left(3x^2 - 2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 ....(1)$$

Now, we will use again Binomial theorem, then

$$\left(3x^2 - 2ax\right)^3$$

$$= {}^3C_0\left(3x^2\right)^3 - {}^3C_1\left(3x^2\right)^2(2ax) + {}^3C_2\left(3x^2\right)(2ax)^2 - {}^3C_3(2ax)^3$$

$$= 27x^6 - 3\left(9x^4\right)(2ax) + 3\left(3x^2\right)\left(4a^2x^2\right) - 8a^3x^3$$

$$= 27x^6 - 54ax^3 + 36a^2x^4 - 8a^3x^3$$

From equation (1) & (2), we get

$$\left(3x^2 - 2ax + 3a^2\right)^3$$

$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^3x + 27a^6$$

$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^3x + 27a^6$$