Chapter – 14 Oscillations

Examples:

14.1 On an average, a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

Answer: The heartbeat frequency:

$= 75/(1\,\text{min})$

$= 75/(60\,\text{s})$

$= 1.25\,\text{s}^{-1}$

$= 1.25\,\text{Hz}$

Time period (T) $= \dfrac{1}{1.25\,s^{-1}}$

$= 0.8\,s$

14.2 Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].

$(i)\ sin\ \omega t\ +\ cos\ \omega t$

Answer:

(i) $sin\ \omega t\ +\ cos\ \omega$ is a periodic function and can also be written as $\sqrt{2}\ sin\left(\omega t\ +\ \dfrac{\pi}{4}\right)$.

$$\therefore w\sqrt{2}\ sin\left(\omega t\ +\ \frac{\pi}{4}\right) = \sqrt{2}\ sin\left(\omega t\ +\ \frac{\pi}{4} + 2\pi\right)$$

$$= \sqrt{2}\ sin\left[\omega\left(t\ +\ \frac{2\pi}{\omega}\right)\ +\ \frac{\pi}{4}\right]$$

$\dfrac{2\pi}{\omega}$ is the periodic time of function.

$(ii)\ sin\ \omega t\ +\ cos\ 2\ \omega t\ +\ sin\ 4\ \omega$

Answer:

(ii) A periodic motion is demonstrated here. Each word represents a periodic function with a different angular frequency, as can be seen. Because period is the shortest amount of time after which a function's value, $sin\ \omega t$ has a period $T_0 = \frac{2\pi}{\omega}$; $cos\ 2\ \omega t$ has a period $\frac{\pi}{\omega} = \frac{T_0}{2}$; and $sin\ 4\ \omega t$ has period $\frac{2\pi}{4\omega} = \frac{T_0}{4}$. The first term's period is a multiple of the last two terms' periods. As a result, T_o is the shortest time interval after which the total of the three terms repeats and thus the sum has a period of $\frac{2\pi}{\omega}$ and is a periodic function.

$(iii)\ e^{-\omega}$

Answer:

(iii) The function $e^{-\omega t}$ is not periodic; it drops monotonically with increasing time and tends to zero as t approaches ∞, therefore its value never repeats.

$(iv)\ log\ (\omega t)$

Answer:

(iv) With time t, the function $\log(\omega t)$ rises monotonically. As a result, it is a non-periodic function that never repeats its value. It should be noticed that as t increases to ∞, $\log(\omega t)$ diverges to ∞. As a result, it can't represent any form of actual displacement.

14.3 Which of the following functions of time represent (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.

a) $sin\ \omega t\ -\ cos\ \omega t$

Answer:

(a) $sin\ \omega t\ -\ cos\ \omega t = sin\ \omega t\ -\ sin\left(\frac{\pi}{2} - \omega t\right)$

$= 2\ cos\left(\frac{\pi}{4}\right) sin\left(\omega t - \frac{\pi}{4}\right)$

$= \sqrt{2}\ sin\left(\omega t - \frac{\pi}{4}\right)$

This function depicts a simple harmonic motion with a period of $T=\frac{2\pi}{\omega}$ and a phase angle of $(\frac{-\pi}{4})$ or $(\frac{7\pi}{4})$ with a period of $T=\frac{2\pi}{\omega}$.

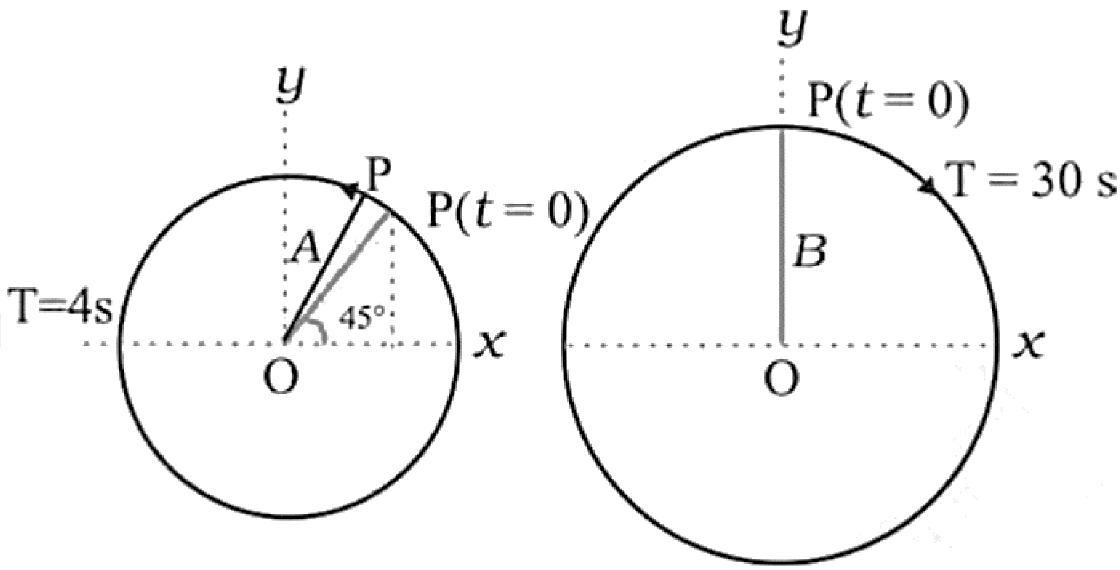
b) $\sin 2\omega t$

Answer:

(b) $\sin^2 \omega t = \frac{1}{2} - \frac{1}{2}\cos 2\omega t$

With a period of $T=\frac{\pi}{\omega}$, the function is periodic. It also symbolises a harmonic motion with the equilibrium point. Instead of zero, it occurs at $\frac{1}{2}$.

14.4 The figure given below depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated in the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.



Answer:

(a) OP makes a $45^0 = \frac{\pi}{4}$ rad angle with the (positive direction of) x-axis at $t = 0$. After time t, it makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x-axis and covers an angle of $\frac{2\pi}{T}t$ in the anticlockwise sense.

The x-axis projection of OP at time t is given by,

$$x(t) = A\cos\left(\frac{2\pi}{T}t + \frac{\pi}{4}\right)$$

For $T = 4s$, $x(t) = A\cos\left(\frac{2\pi}{4}t + \frac{\pi}{4}\right)$ It is an amplitude A of SHM with a period of 4 s and an initial phase of $\frac{\pi}{4}$.

(b) When $t = 0$ is reached, OP forms a $90^0 = \frac{\pi}{2}$ angle with the x-axis. It reaches an angle of $\frac{2\pi}{T}t$ in the clockwise sense and makes an angle of $\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$ with the x-axis after a period t. The x-axis projection of OP at time t is given by

$$x(t) = B\cos\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$$

$$= B\sin\left(\frac{2\pi}{T}t\right)$$

For $T = 30s$, $x(t) = B\sin\left(\frac{2\pi}{T}t\right)$

$\therefore x(t) = B\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right)$ and comparing with equation $x(t) = A\cos(\omega t + \varphi)$. This corresponds to a SHM with amplitude B, period 30 s, and an initial phase of $\frac{-\pi}{2}$.

14.5 A body oscillates with SHM according to the equation (in SI units.

$$x = 5\cos\left[2\pi t + \frac{\pi}{4}\right]$$ **At t = 1.5 s, calculate the**

(a) displacement,

Answer:

The body's angular frequency ω is $2\pi\ s^{-1}$ and its temporal period is $T = 1s$.

At $t = 1.5\ s$

(a) $displacement = (5.0\ m)\ cos\left[\left(2\pi\ s^{-1}\right) \times 1.5\ s + \frac{\pi}{4}\right]$

$= (5.0\ m)\ cos\left[\left(3\pi + \frac{\pi}{4}\right)\right]$

$= -5.0 \times 0.707\ m$

$= -3.535\ m$

(b) speed

Answer:

(b) Using equation $v(t) = -\omega A\ sin\ (\omega t + \varphi)$

Speed of body = $-\ (5.0\ m)(2\pi\ s\ -1)\ sin\left[\left(2\pi\ s^{-1}\right)\times 1.5\ s + \frac{\pi}{4}\right]$

$= -\ (5.0\ m)(2\pi\ s - 1)\ sin\left[\left(3\pi + \frac{\pi}{4}\right)\right]$

$= 10\pi \times 0.707\ m\ s^{-1}$

$= 22\ m\ s^{-1}$

(c) Acceleration of the body.

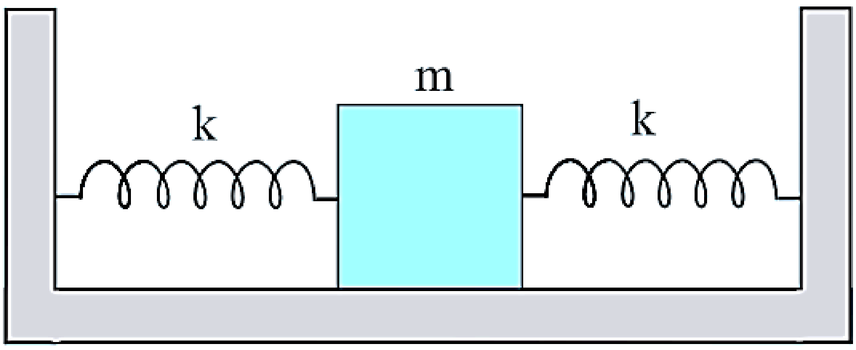
Answer:

(c) Using equation $v(t) = \frac{d}{dx}x(t)$

Acceleration of body $= -\left(2\pi\ s^{-1}\right)^2 \times displacement = -\ \left(2\pi\ s^{-1}\right)^2 \times (-3.535\ m)$

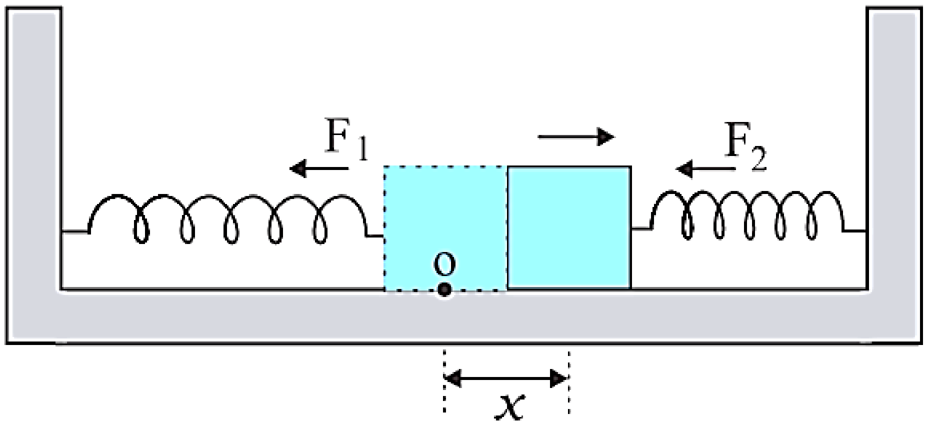
$= 140\ m\ s^{-2}$

14.6 Two identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown in figure. Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations.



Answer:

As indicated in figure, move the mass to the right side of the equilibrium position by a little distance x.



The spring on the left side is stretched by x length and the spring on the right side is compressed by the same length in this case. As a result, the forces acting on the mass are,

$F_1 = -k\,x$ (Force applied on the left side by the spring, attempting to draw the mass back to its normal position)

$F_2 = -k\,x$ (Force applied on the right side by the spring, attempting to push the mass towards the centre)

The net force exerted on the mass, F, is then calculated as follows: $F = -2kx$

As a result, the force exerted on the mass is proportionate to the displacement and directed towards the mean position; thus, the mass' motion is simple harmonic.

Time period of oscillations is: $T = 2\pi\sqrt{\frac{m}{2k}}$.

14.7 A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m–1. The block is pulled to a distance x = 10 cm from its equilibrium position at x = 0 on a frictionless surface from rest at t = 0. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.

Answer:

SHM is executed by the block, and its angular frequency, as determined by Eq.

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{50\,N\,m^{-1}}{1\,kg}}$$

$$= 7.07\,rad\,s^{-1}$$

The displacement of the object at any given time t is then given by,

$$x(t) = 0.1\,cos\,(7.07t)$$

As a result, we have $0.05 = 0.1\,cos\,(7.07t)$ or $cos\,(7.07t) = 0.5$ when the particle is 5 cm away from the mean position and hence

$$sin\,(7.07t) = \frac{\sqrt{3}}{2} = 0.866$$

The velocity of the block at x = 5 cm is $= 0.1 \times 7.07 \times 0.866\,m\,s^{-1}$

$$= 0.61\,m\,s^{-1}$$

K.E of the block $= \frac{1}{2}mv^2$

$$= \frac{1}{2}\left[1kg \times \left(0.6123\,m\,s^{-1}\right)^2\right]$$

$$= 0.19\,J$$

P.E of the block $= \frac{1}{2}kx^2$

$$= \frac{1}{2}\left(50\,N\,m^{-1} \times 0.05\,m \times 0.05\,m\right)$$

$$= 0.0625\,J$$

Total energy of block at x=5cm $= K.E + P.E$

$$= 0.25\,J$$

We also know that K.E. is zero at maximum displacement; hence the total energy of the system is equal to P.E. Therefore, total energy is

$$= \frac{1}{2}\left(50\,N\,m^{-1} \times 0.1\,m \times 0.1\,m\right)$$

$$= 0.25\,J$$

At a distance of 5 cm, this equals the sum of the two energies. This is consistent with the energy conservation principle.

14.8 A 5 kg collar is attached to a spring of spring constant 500 N m–1. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate

(a) the period of oscillation,

Answer:

(a) The oscillations of period are given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5.0\,kg}{500\ N\,m^{-1}}}$$

$$= \left(\frac{2\pi}{10}\right)s$$

$$= 0.63\ s$$

(b) the maximum speed and

Answer:

(b) The velocity of the collar executing SHM is calculated as follows:

$$v(t) = -A\omega \sin\left(\omega t + \varphi\right)$$

The maximum speed is given by; $v_m = A\omega$

$$= 0.1 \times \sqrt{\frac{k}{m}}$$

$$= 0.1 \times \sqrt{\frac{500\,N\,m^{-1}}{5\,kg}}$$

$= 1\ m\ s^{-1}$ and occurs at x=0.

(c) maximum acceleration of the collar.

Answer:

(c) The collar's acceleration at a displacement x (t) from equilibrium is given by

$$a\ (t) = -\omega^2\ x(t)$$

$$= -\frac{k}{m}x(t)$$

Therefore, maximum acceleration is $a_{\max} = \omega^2\ A$

$$= \frac{500\,Nm^{-1}}{5\,kg} \times 0.1m$$

$= 10\ m\ s^{-2}$ occurs at extremities.

14.9 What is the length of a simple pendulum, which ticks seconds?

Answer:

A basic pendulum's time period is given by, $T = 2\pi\sqrt{\frac{L}{g}}$

From this, we get $L = \frac{gT^2}{4\pi^2}$

A simple pendulum that ticks seconds has a time period of 2 seconds. As a result,for $g = 9.8\ m\ s^{-2}$ and $T = 2\ s$ L is:

$$= \frac{9.8\left(m\ s^{-2}\right) \times 4\left(s^2\right)}{4\pi^2}$$

$$= 1\ m$$

14.10 For the damped oscillator shown in Fig. 14.19, the mass m of the block is 200 g, $\mathbf{k = 90\ N\ m^{-1}}$ and the damping constant b is $\mathbf{40\ g\ s^{-1}}$. Calculate (a) the period of oscillation, (b) time taken for its amplitude of vibrations to drop to half of its initial value, and (c) the time taken for its mechanical energy to drop to half its initial value.

Answer:

(a) Here, $km = 90 \times 0.2 = 18\ kg\ Nm^{-1}$

$\therefore \sqrt{km} = 4.243\ kg\ s^{-1}$ and $b = 0.04\ kg\ s^{-1}$

$\therefore b \ll \sqrt{km}$

Therefore, $T = 2\pi\sqrt{\frac{m}{k}}$

$$= 2\pi\sqrt{\frac{0.2\,kg}{90\ N\,m^{-1}}}$$

$$= 0.3\ s$$

(b) $T_{\frac{1}{2}}$ is the time it takes for the amplitude to drop to half of its initial value.

$$\therefore T_{\frac{1}{2}} = \frac{\ln\frac{1}{2}}{\frac{b}{2m}}$$

$$= \frac{0.693}{40} \times 2 \times 200\ s$$
$$= 6.93\ s$$

(c) The time, $t_{\frac{1}{2}}$, for its mechanical energy to reduce to half its initial value. We can deduce the following from this equation:

$$\frac{E\left(t_{\frac{1}{2}}\right)}{E(0)} = exp\left(\frac{-bt_{\frac{1}{2}}}{m}\right)$$

$$\therefore \frac{1}{2} = \exp\left(\frac{-bt_{\frac{1}{2}}}{m}\right)$$

$$\therefore \ln\left(\frac{1}{2}\right) = -\left(\frac{bt_{\frac{1}{2}}}{m}\right)$$

$$\therefore t_{\frac{1}{2}} = \frac{0.639}{40\ gs^{-1}} \times 200\,g$$

$$= 3.46\ s$$

This is only half of the amplitude decay period. This is not surprising, because energy is proportional to the square of the amplitude, according to Equations:

$x(t) = A\,e^{-b\,t/2m}\,cos\,(\omega' t + \varphi)$ and $E(t) = \frac{1}{2} k\,A^2\,e^{\frac{-bt}{m}}$. It's worth noting that the exponents of the two exponentials differ by a factor of two.

Exercises

14.1 Which of the following examples represent periodic motion?

a) A swimmer completing one (return) trip from one bank of a river to the other and back.

Answer: The motion of the swimmer is not periodic. Swimming between the banks of a river is a back-and-forth motion. It does not, however, have a set duration. This is because the swimmer's back and forth voyage may not take the same amount of time.

(b) A freely suspended bar magnet displaced from its N-S direction and released.

Answer: When a freely suspended magnet is displaced from its N-S orientation and released, it moves in a predictable pattern. This is due to the fact that the magnet oscillates about its position over a set length of time.

(c) A hydrogen molecule rotating about its centre of mass.

Answer: When a hydrogen molecule spins about its mass centre, it returns to the same location after an equal amount of time. This type of motion is periodic.

(d) **An arrow released from a bow.**

Answer: When an arrow is fired from a bow, it simply moves forward. It does not go in the opposite direction. As a result, this motion isn't periodic.

14.2 Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?

a) the rotation of earth about its axis.

Answer: Earth returns to the same position in equal intervals of time during its rotation around its axis. As a result, it's a periodic motion. This motion, however, is not simple harmonic. This is due to the fact that the earth does not rotate on its axis

(b) motion of an oscillating mercury column in a U-tube.

Answer: A simple harmonic is an oscillating mercury column in a U-tube. This is due to the fact that the mercury moves back and forth on the same path, around the same fixed place, during a set length of time.

(c) Motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.

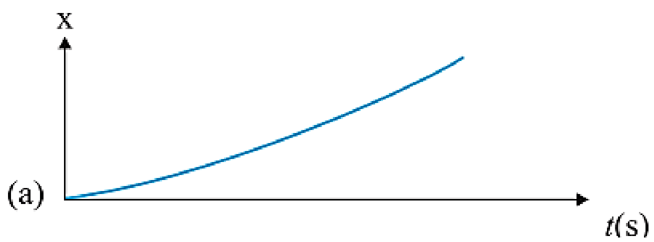
Answer: When the ball is released, it bounces around the bowl's lowermost point. In addition, the ball repeatedly returns to its original place in the same amount of time. As a result, its motion is both periodic and simple harmonic.

(d) general vibrations of a polyatomic molecule about its equilibrium position.

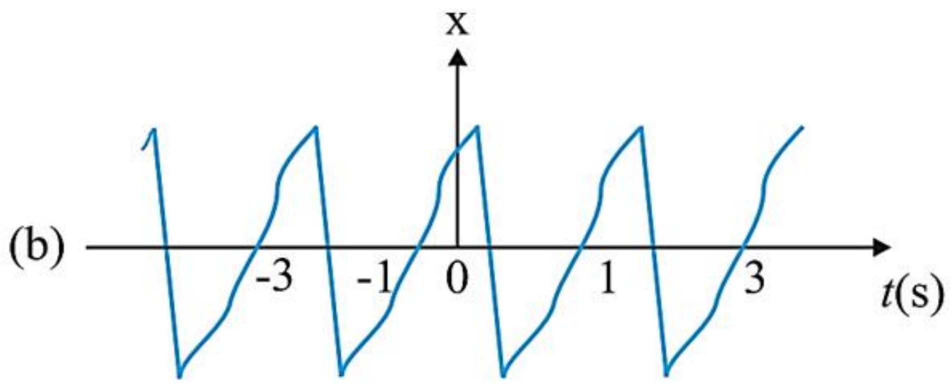
Answer: The natural oscillation frequencies of a polyatomic molecule are numerous. Its vibration is the result of the superposition of a number of distinct molecules' separate simple harmonic movements. As a result, it is periodic rather than simple harmonic.

(b) and (c) are SHMs, (a) and (d) are periodic, but not SHMs.

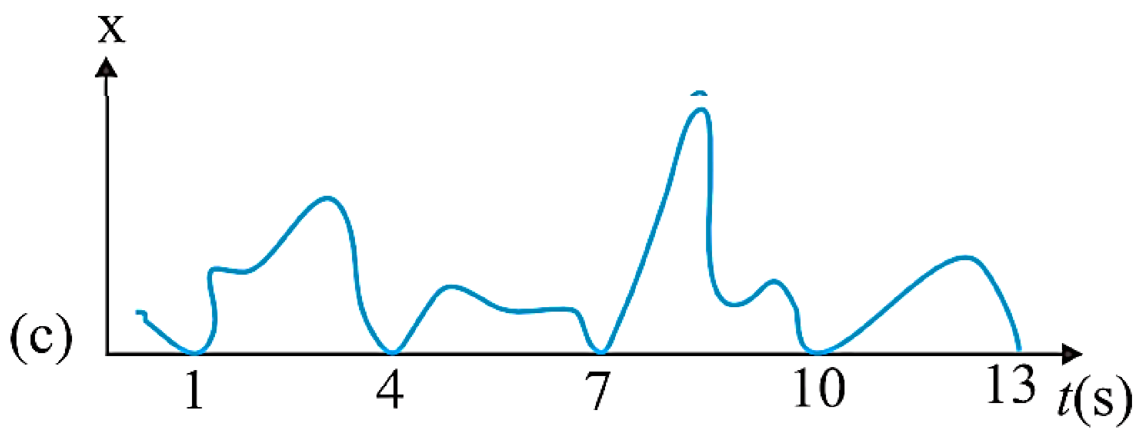
14.3 Figure depicts four x-t plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion)?



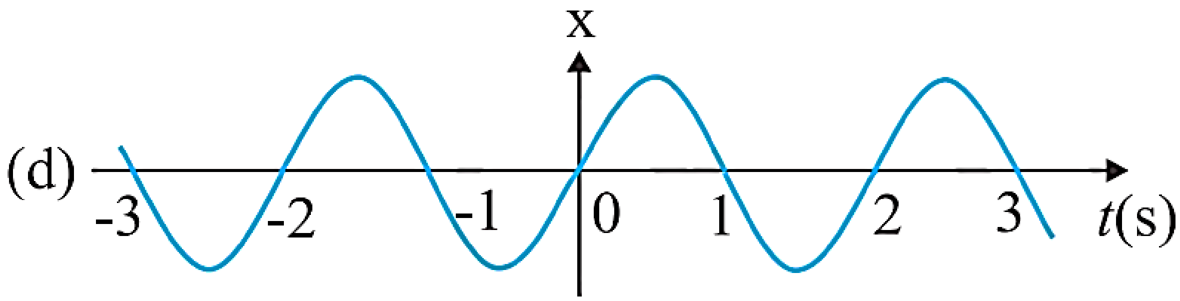
Answer: It isn't a periodic motion. This is a unidirectional, uniform linear motion. In this scenario, there is no repeat of motion.



Answer: The particle's motion repeats itself after 2 seconds in this scenario. As a result, it is a periodic motion with a period of 2 seconds.



Answer: It isn't a regular motion. This is due to the fact that the particle only repeats the motion in one position. The complete motion of the particle must be repeated in equal intervals of time for a periodic motion to occur.



Answer: The particle's motion repeats itself after 2 seconds in this scenario. As a result, it is a periodic motion with a period of 2 seconds.

(b) and (d) are periodic

14.4 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non periodic motion? Give period for each case of periodic motion (ω is any positive constant):

$(a)\ sin\ \omega t\ -\ cos\ \omega t$

Answer: $\sin \omega t - \cos \omega t = \sqrt{2}\left[\frac{1}{\sqrt{2}} \sin \omega t - \frac{1}{\sqrt{2}} \cos \omega t\right]$

$$= \sqrt{2}\left[\sin \omega t \times \cos \frac{\pi}{4} - \cos \omega t \times \sin \frac{\pi}{4}\right]$$

$$= \sqrt{2} \sin\left(\omega t - \frac{\pi}{4}\right)$$

This function represents SHM in the following format: $a \sin(\omega t + \phi)$

Its period is: $\frac{2\pi}{\omega}$. Hence, it's periodic and not SHM.

$(b)\ sin^3\ \omega t$

Answer: Individually, the terms $\sin \omega t$ and $\sin \omega t$ reflect simple harmonic motion (SHM).The superposition of two SHM, on the other hand, is periodic rather than simple harmonic.

$(c)\ 3\ cos\left(\frac{\pi}{4}\ -\ 2\omega t\right)$

Answer: Because it may be stated in the form: $a\sin(\omega t+\phi)$.This function describes simple harmonic motion.

Its period is $\frac{2\pi}{2\omega}=\frac{\pi}{\omega}$ so it's periodic and not SHM.

$(d)\ \cos\omega t+\cos 3\omega t+\cos 5\omega t$

Answer: $\cos\omega t+\cos 3\omega t+\cos 5\omega t$. SHM is represented by each unique cosine function. The superposition of three simple harmonic motions, on the other hand, is periodic, but it is not simple harmonic. So, it is non periodic motion.

$(e)\ \exp\left(-\omega^2 t^2\right)$

Answer: $\exp\left(-\omega^2 t^2\right)$. Exponential functions do not repeat. Therefore, it is non-periodic.

$(f)\ 1+\omega t+\omega^2 t^2$

Answer: $1+\omega t+\omega^2 t^2$ is non-periodic.

14.5 A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

a) At the end A,

Answer: Zero, Positive, Positive

b) At the end B,

Answer: Zero, Negative, Negative

c) At the mid-point of AB going towards A,

Answer: Negative, Zero, Zero

d) At 2 cm away from B going towards A,

Answer: Negative, Negative, Negative

e) At 3 cm away from A going towards B,

Answer: Zero, Positive, Positive

f) At 4 cm away from B going towards A.

Answer: Negative, Negative, Negative

14.6 Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?

$(a)\ a = 0.7x$

$(b)\ a = -200x^2$

$(c)\ a = -10x$

$(d)\ a = 100x^3$

Answer:

$$F = -kx$$
$$\therefore ma = -kx$$
$$\therefore a = -\frac{k}{m}x$$

Where, F = force

X = displacement

a = acceleration

k = constant

Among the equations which are given only $a = -10x$ is written in the form $\frac{k}{m} = 10$.

Therefore, it represents SHM.

14.7 The motion of a particle executing simple harmonic motion is described by the displacement function, $x(t) = A\cos(\omega t + \varphi)$. If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is $\pi\ s^{-1}$. If instead of the cosine function, we choose the sine function to describe the SHM: x = B sin (ωt + α), what are the amplitude and initial phase of the particle with the above initial conditions.

Answer:

At t = 0;

Displacement, x = 1 cm

Initial velocity, $v = \omega\ cm/sec$.

Angular frequency, $\omega = \pi\ rad/s^{-1}$

$$x(t) = A\cos(\omega t + \phi)$$
$$1 = A\cos(\omega \times 0 + \phi) = A\cos\phi$$
$$A\cos\phi = 1 \qquad (i)$$

$$Velocity, v = \frac{dy}{dx}$$

$$\omega = -A\omega\sin\left(\omega t + \phi\right)$$
$$1 = -A\sin\left(\omega\times 0 + \phi\right) = -A\sin\phi$$
$$A\sin\phi = -1 \qquad (ii)$$

Squaring and adding equations (i) and (ii) we get,

$$A^2\left(\sin^2\phi + \cos^2\phi\right) = 1 + 1$$
$$A^2 = 2$$
$$\therefore A = \sqrt{2}\ cm$$

Dividing eq.(ii) by (i) we get:

$$\tan\phi = -1$$
$$\therefore \phi = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots\ldots\ldots\ldots$$

SHM is given as: $x = B\sin\left(\omega t + \alpha\right)$

$$\therefore 1 = B\sin\left(\omega \times 0 + \alpha\right)$$
$$B\sin\alpha = 1 \qquad (iii)$$

Velocity, $v = \omega B\cos\left(\omega t + \alpha\right)$

$$\therefore \pi = \pi B\sin\alpha$$
$$B\sin\alpha = 1 \qquad (iv)$$

Squaring and adding equations (iii) and (iv), we get:

$$B^2\left[\sin^2\alpha + \cos^2\alpha\right] = 1 + 1$$
$$B^2 = 2$$
$$\therefore B = \sqrt{2}\ cm$$

Dividing equations (iii) by equation (iv):

$$\frac{B\sin\alpha}{B\cos\alpha} = \frac{1}{1}$$

$$\tan\alpha = 1 = \tan\frac{\pi}{4}$$

$$\therefore \alpha = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots\ldots\ldots\ldots$$

14.8 A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What isthe weight of the body?

Answer:

M = 50kg

Maximum displacement of the spring = Length of the scale, l = 20 cm = 0.2 m

Time period, T = 0.6 s

Maximum force exerted on the spring, $F = Mg$

g = acceleration due to gravity

$$= 9.8\ \frac{m}{s^2}$$

F = 50 × 9.8 = 490

Therefore, spring constant $k = \frac{F}{l} = \frac{490}{0.2} = 2450\,Nm^{-1}$

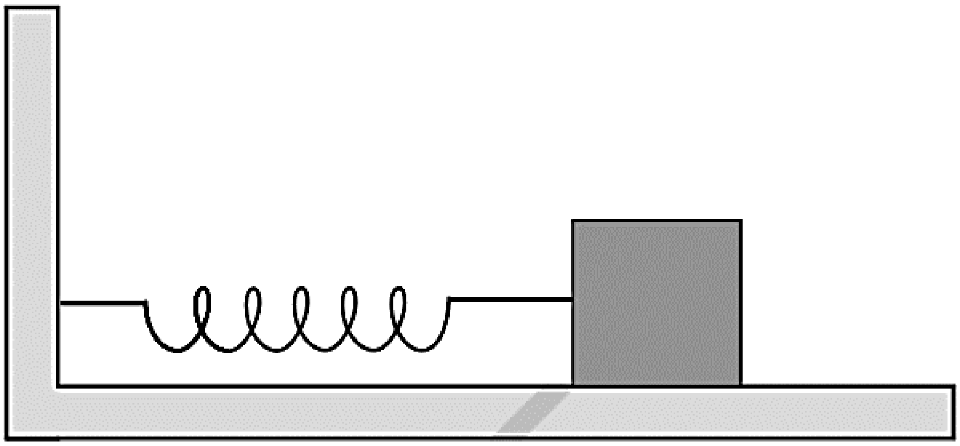
Time period, $T = 2\pi\sqrt{\frac{m}{k}}$

$$\therefore m = \left(\frac{T}{2\pi}\right)^2 \times k = \left(\frac{0.6}{2 \times 3.14}\right)^2 \times 2450 = 22.36\ kg$$

Weight of body = $mg = 22.36 \times 9.8 = 219.167\,N$

Hence, weight of body is 219N.

14.9 A spring having with a spring constant 1200 N m–1 is mounted on a horizontal table as shown in figure. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.



Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

Answer:

$k = 1200 \text{ N m}^{-1}$

$m = 3kg$

$A = 2.0 \text{ cm} = 0.02 \text{ cm}$

Frequency $v = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$\therefore v = \frac{1}{2 \times 3.14}\sqrt{\frac{1200}{3}} = 3.18 \; ms^{-1}$$

Hence, the frequency is 3.18 m/s.

Now, $a = \omega^2 A$

Where, $\omega = Angular \; frequncy = \sqrt{\frac{k}{m}}$

$A = \max \; displacemnt$

$$\therefore a = \frac{k}{m}A = \frac{1200 \times 0.02}{3} = 8 \; ms^{-1}$$

Maximum velocity, $V_{\max} = A\omega$

$= A\sqrt{\frac{k}{m}} = 0.02 \times \sqrt{\frac{1200}{3}} = 0.4 \; m/s$. Hence, maximum velocity is 0.4 m/s.

14.10 In Exercise 14.9, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0),the mass is

(a) at the mean position,

(b) at the maximum stretched position, and

(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, infrequency, in amplitude or the initial phase?

Answer:

x = 2sin 20t
x = 2cos 20t
x = $-2\cos 20t$

Here,

A = 2.0 cm
k = 1200 N m^{-1}
m = 3 kg

Angular frequency, $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1200}{3}} = \sqrt{400} = 20\, rad\, s^{-1}$

When mass is at rest position, starting phase is 0.

Displacement, $x = A\sin\omega t = 2\sin 20t$

At max position, mass is toward extreme right. So, initial phase is $\dfrac{\pi}{2}$.

Displacement,

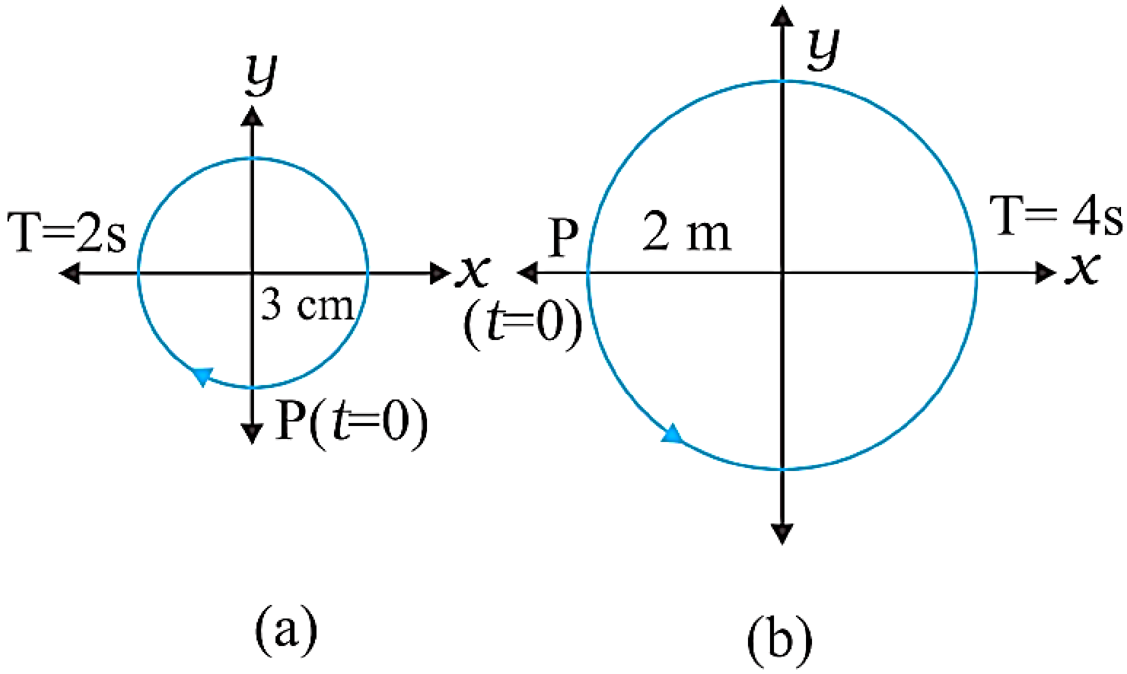
$$x = A\sin\left(\omega t + \frac{\pi}{2}\right) = 2\sin\left(20t + \frac{\pi}{2}\right) = 2\cos 20t$$

At max compressed position, mass is toward extreme left. So, initial phase is $\dfrac{3\pi}{2}$.

Displacement, $x = A\sin\left(\omega t + \dfrac{3\pi}{2}\right) = 2\sin\left(20t + \dfrac{3\pi}{2}\right) = -2\cos 20t$

Therefore, the function has same frequency $\left(\dfrac{20}{2\pi} Hz\right)$ and amplitude (2 cm), but different initial phases $\left(0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

14.11 Figures correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and thesense of revolution (i.e., clockwise or anti-clockwise) are indicated on each figure.



Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

Answer:

Time period, T = 2

Amplitude, A = 3 cm

At time, t = 0, the radius vector OP makes an angle $\frac{\pi}{2}$ with the positive x-axis, i.e., $\phi = +\frac{\pi}{2}$

Therefore, displacement equation:

$$x = A\cos\left[\frac{2\pi t}{T} + \phi\right] = 3\cos\left(\frac{2\pi t}{2} + \frac{\pi}{2}\right) = -3\sin \pi t\, cm$$

Now,

Time period, T = 4 s

Amplitude, a = 2 m

At time t = 0, OP makes an angle π with the x-axis, in the anticlockwise direction, i.e., $\phi = +\pi$

Therefore, displacement equation:

$$x = a\cos\left(\frac{2\pi t}{T} + \phi\right) = 2\cos\left(\frac{2\pi t}{T} + \pi\right)$$

$$= -2\cos\left(\frac{\pi}{2}t\right)m$$

14.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t =0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

$(a)\ x = -2\ sin\left(3t + \frac{\pi}{3}\right)$

Answer: $x = -2\sin\left(3t + \frac{\pi}{3}\right) = +2\cos\left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right) = 2\cos\left(3t + \frac{5\pi}{6}\right)$

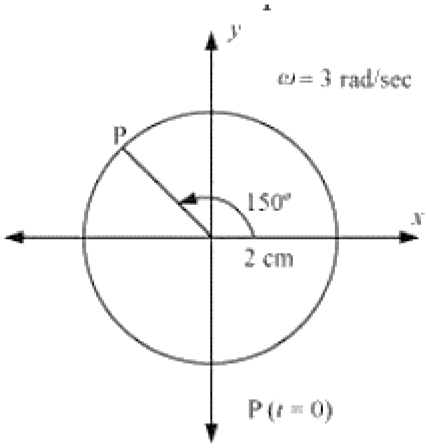
If we compare this equation with standard SHM equation then we get:

$Amplitude = 2\, cm$

$Phase\, angle\ \phi = \frac{5\pi}{6} = 150^0$

$Angular\, velocity, \omega = \frac{2\pi}{T} = 3\, rad/\sec$

The motion is as follows:



$(b)\ x = \cos\left(\frac{\pi}{6} - t\right)$

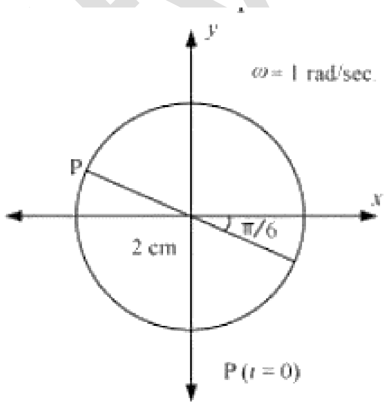
Answer: If we compare this equation with standard SHM equation then we get:

$$Amplitude = 1cm$$

$$Phase\,angle\ \phi = -\frac{\pi}{6} = -30^0$$

$$Angular\ velocity, \omega = \frac{2\pi}{T} = 1 rad/\sec$$

The motion is as follows:



$(c)\ x = 3\sin\left(2\pi t + \frac{\pi}{4}\right)$

Answer:

$$x = 3\sin\left(2\pi t + \frac{\pi}{4}\right) = -3\cos\left[\left(2\pi t + \frac{\pi}{4}\right) + \frac{\pi}{2}\right] = -3\cos\left(2\pi t + \frac{3\pi}{4}\right)$$

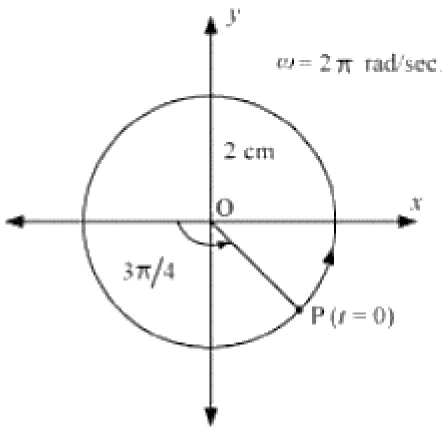
If we compare this equation with standard SHM equation then we get:

$Amplitude = 3cm$

$Phase\, angle\ \phi = \frac{3\pi}{4} = 135^0$

$Angular\ velocity, \omega = \frac{2\pi}{T} = 2\pi\, rad / \sec$

The motion is as follows:



$(d)\ x = 2\ cos\ \pi t$

Answer:

x = 2 cos πt

If we compare this equation with standard SHM equation then we get:

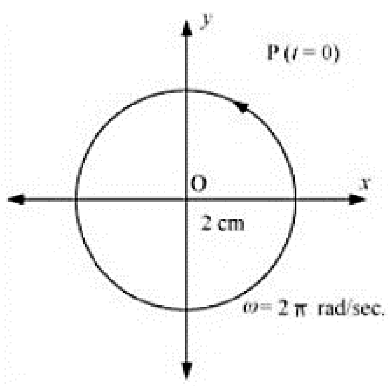
Amplitude $= 2cm$

Phase angle $\phi = 0$

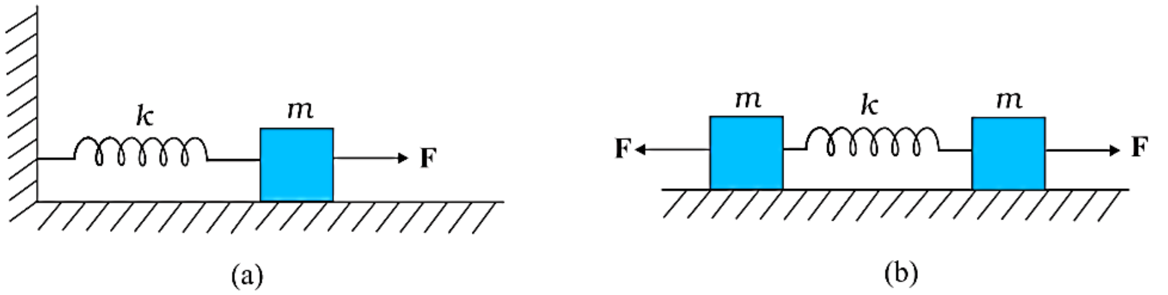
Angular velocity

$\omega = \pi\, rad / \sec$

The motion is as follows:



14.13 Figure shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. (b) is stretched by the same force F.



a) **What is the maximum extension of the spring in the two cases? (b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case?**

Answer:

<u>One block system:</u>

$$F = kl$$

Hence, max extension in spring is $l = \frac{F}{k}$.

<u>Two block system:</u>

Displacement, $x = \frac{l}{2}$

$$Force, F = +2\ kx = 2k\frac{l}{2}$$

$$\therefore l = \frac{F}{k}$$

<u>One block system:</u>

$$F = ma = m\frac{d^2y}{dt^2}$$

$$\therefore m\frac{d^2y}{dt^2} = -kx$$

Direction of force is opposite to the direction of displacement so it’s negative.

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x = -\omega^2 x$$

$$where,\ \omega^2 - \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}\ ,\omega\ is\ an\ angular\ \ frequency$$

Time period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{\dfrac{k}{m}}} = 2\pi\sqrt{\dfrac{m}{k}}$

<u>Two block system:</u>

$$F = m\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} = -2kx$$

Direction of force is opposite to the direction of displacement so it’s negative.

$$\frac{d^2x}{dt^2} = -\left(\frac{2k}{m}\right)x = -\omega^2 x$$

Where,

$$\omega = \sqrt{\frac{2k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2k}}$$

14.14 The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed?

Answer:

Frequency of piston $= 200\,rad\,/\min$

Stroke $= 1.0m$

Amplitude, $A=\frac{1.0}{2}=0.5m$

Max speed is given by, $V_{\max} = A\omega$

$=200\times0.5=100\,m/\min$

14.15The acceleration due to gravity on the surface of moon is 1.7 m s 2. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is $9.8\ \mathrm{m\ s^{-2}}$)

Answer:

Acceleration due to gravity on the moon, $= 1.7\ \mathrm{m\ s^{-2}}$

Acceleration due to gravity on the Earth $\mathrm{g = 9.8\ m\ s^{-2}}$

Time period of a pendulum on Earth $\mathrm{T = 3.5\ s}$

$$\mathrm{T} = 2\pi\sqrt{\frac{l}{g}},\ l = length\ of\ \ pendulum$$

$$\therefore l=\frac{T^2}{(2\pi)^2}\times g$$

$$=\frac{(3.5)^2}{4\times(3.14)^2}\times 9.8\ m$$

The length of pendulum remains constant.

On surface of the moon, Time period:

$$T^{'} = 2\pi\sqrt{\frac{l}{g^{'}}}$$

$$=2\pi\sqrt{\frac{\frac{(3.5)^2}{4\times(3.14)^2}\times 9.8}{1.7}}=8.4s$$

Hence, the time period is 8.4 s of pendulum on the surface of moon.

14.16 Answer the following questions:

(a) Time period of a particle in SHM depends on the force constant k and mass m of the particle: $T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

(c) A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

(d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?

Answer:

Time period $T = 2\pi\sqrt{\frac{m}{k}}$

For simple pendulum, $k \propto m$

$$\frac{m}{k} = \text{Constant}$$

Hence, simple pendulum is independent of the mass of the bob.

Here, restoring force $F = -mg\sin\theta$

Where, F= restoring force

m = mass of bob

g = acceleration due to gravity

θ = angle of displacement

For small θ, $\sin\theta \simeq \theta$

Hence, Time period $T = 2\pi\sqrt{\frac{l}{g}}$

The time displayed on a man's wristwatch as he falls from the top of a skyscraper remains unaffected by the fall. Because a wristwatch does not operate on the same basis as a basic pendulum, it is unaffected by gravity's acceleration during free fall. Its operation is reliant on spring motion.

The acceleration of a simple pendulum mounted in a cabin while it falls freely under gravity is zero. As a result, the oscillation frequency of this simple pendulum is zero.

14.17 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position,what will be its time period?

Answer:

The bob of the simple pendulum will be subjected to gravity's acceleration as well as the centripetal acceleration supplied by the car's circular motion.

Acceleration due to gravity = g

Centripetal acceleration $= \frac{v^2}{R}$

Effective acceleration (a_{eff}) $a_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}$

Time period T, $T = 2\pi\sqrt{\frac{l}{a_{eff}}}$

$$T = 2\pi\sqrt{\frac{l}{g^2 + \frac{v^4}{R^2}}}$$

14.18 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_1. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period $T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$ where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

Answer:

Area of cork = A

Height of cork = h

Density of liquid = ρ_1

Density of cork = ρ

Equilibrium: Weight of cork = weight of liquid displaced by floating cork

Here, Up-thrust = Restoring force

Cork displayed slightly by x.

F = weight of extra water displaced

$$F = -(volume \times density \times g)$$

$$Volume(v) = Ax$$

$$\therefore F = -Ax\rho_1 g \qquad ....(i)$$

Now,

$F = kx$

$\therefore k = \frac{F}{x} = -A\rho_1 g \qquad ...(ii)$

Time period, $T = 2\pi\sqrt{\frac{m}{k}} \qquad ...(iii)$

Where, m = mass of cork

= volume x density

= Base area of cork x height of cork x density

$= Ah\rho$

Hence, Time period $T = 2\pi\sqrt{\frac{Ah\rho}{A\rho_1 g}} = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$

14.19One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Answer:

Area of cross section $= A$

Density $= \rho$

Acceleration due to gravity $= g$

Restoring force, F = weight of mercury

$F = -(volume \times density \times g)$

$\therefore F = -(A \times 2h \times \rho \times g) = -2Ah\rho g = -k \times \text{displacement in one arm}\,(h)$

Where, height of mercury in two arms is 2h

$k = -\frac{F}{h} = 2A\rho g$

$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{2A\rho g}}$, where m = mass of mercury

Let, l = length of mercury

Mass of mercury = volume x density

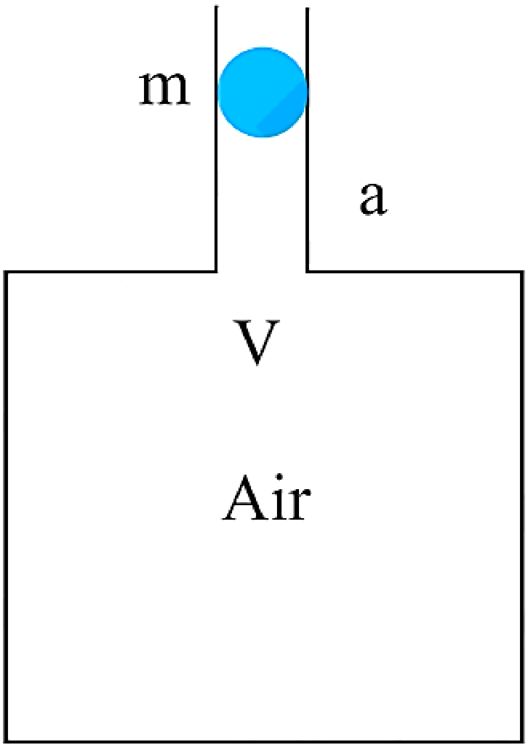
$$= B = \frac{stress}{strain} = \frac{-p}{\frac{ax}{V}}$$

$$\therefore T = 2\pi\sqrt{\frac{Al\rho}{2A\rho g}} = 2\pi\sqrt{\frac{l}{2g}}$$

As a result, the mercury column moves in a simple harmonic motion with a time period of $2\pi\sqrt{\frac{l}{2g}}$.

Additional Exercises:

14.20 An air chamber of volume V has a neck area of cross section into which a ball of mass m just fits and can move up and down without any friction. Show that when the ball is pressed down a little and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure volume variations of air to be isothermal.



Answer:

Air-chamber volume = V

Cross-section area of neck = a

Mass of ball = m

Pressure inside chamber = atmospheric pressure

Let ball depressed by x units. There would be a decrease in volume and an increase in pressure inside the chamber as a result of this depression.

$$\text{Decrease in volume}, \Delta V = ax$$

$$\text{Volumetric strain} = \frac{\text{change in volume}}{\text{original volume}}$$

$$\therefore \frac{\Delta V}{V} = \frac{ax}{V}$$

Therefore, bulk modulus $B = \frac{stress}{strain} = \frac{-p}{\frac{ax}{V}}$

Stress is defined as an increase in pressure in this circumstance. The negative sign shows that as volume decreases, pressure rises.

$$p = \frac{-Bax}{V}$$

Restoring force, $F = p \times a$

$$= \frac{-Bax}{V} \times a$$

$$= \frac{-Ba^2x}{V} \quad \ldots\ldots(i)$$

In SHM, restoring force $F = -kx \quad \ldots.(ii)$

Comparing eq. (i) and (ii) $k = \frac{Ba^2}{V}$

And Time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{Vm}{Ba^2}}$

14.21 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50%during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

Answer:

Mass of automobile, m = 3000Kg

Displacement, x=15 cm = 0.15 m

There are four springs running parallel to the support of the automobile's bulk.

Restoring force, $F = -4kx = mg$

Time period, $T = 2\pi\sqrt{\frac{m}{4k}}$

$$\therefore k = \frac{mg}{4x} = \frac{3000 \times 10}{4 \times 0.15} = 5000 = 5 \times 10^4 \ N/m$$

Mass supported by each wheel, $M = \frac{3000}{4} = 750\,kg$

For damping factor, displacement $x = x_0 e^{\frac{-bt}{2M}}$

The oscillation amplitude decreases by 50%

$$\therefore x = \frac{x_0}{2}$$

$$\frac{x_0}{2} = x_0 e^{\frac{-bt}{2M}}$$

$$\log_e 2 = \frac{bt}{2M}$$

$$\therefore b = \frac{2M \log_e 2}{t}$$

Where, time period $t = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{3000}{4 \times 5 \times 10^4}} = 0.7691\ s$

$$\therefore b = \frac{2 \times 750 \times 0.693}{0.7691} = 1351.58\ kg/s$$

Therefore, spring’s damping constant is 1351.58 kg/s.

14.22 Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.

Answer:

The displacement of a particle performing SHM at time t is given by the equation: $x = A\sin\omega t$

Where,

$amplitude\ of\ oscillation = A$

$Angular\ frequency(\omega) = \sqrt{\frac{k}{m}}$

Velocity of particle, $v = \frac{dy}{dx} = A\omega\cos\omega t$

Kinetic energy of particle is: $E_k = \frac{1}{2}Mv^2 = \frac{1}{2}MA^2\omega^2\cos^2\omega t$

Potential energy of particle is: $E_p = \frac{1}{2}kx^2 = \frac{1}{2}M\omega^2A^2\sin^2\omega t$

For average kinetic energy:

$$
\begin{aligned}
(E_k)_{Avg} &= \frac{1}{T}\int_0^T E_k\,dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2}MA^2\omega^2\cos^2\omega t\,dt \\
&= \frac{1}{2T}MA^2\omega^2\int_0^T \frac{(1+\cos 2\omega t)}{2}\,dt \\
&= \frac{1}{4T}MA^2\omega^2\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T \\
&= \frac{1}{4T}MA^2\omega^2(T) \\
&= \frac{1}{4}MA^2\omega^2 \qquad ...(i)
\end{aligned}
$$

Average potential energy:

$$\left(E_p\right)_{Avg} = \frac{1}{T}\int_0^T E_p\ dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2} M\omega^2 A^2 \sin^{2\omega t}\ dt$$

$$= \frac{1}{2T} M\omega^2 A^2 \int_0^T \frac{(1-\cos 2\omega t)}{2}\ dt$$

$$= \frac{1}{4T} M\omega^2 A^2 \left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T} M\omega^2 A^2 (T)$$

$$= \frac{M\omega^2 A^2}{4} \qquad ...(ii)$$

The average kinetic energy for a given time period is equal to the average potential energy for the same time period, according to equations (i) and (ii).

14.23 A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation J = –α θ, where J is the restoring couple and θ the angle of twist).

Answer:

Mass of circular disc, m = 10 kg

Radius of disc, r = 15 cm = 0.15 cm

Time period T= 1.5 s

Moment of Inertia of disc $I = \frac{1}{2}mr^2$

$$= \frac{1}{2} \times (10) \times (0.15)^2$$

$$= 0.1125\ kg\, m^2$$

Time period, $T = 2\pi\sqrt{\frac{I}{\alpha}}$ where α is the torsional constant

$$\therefore \alpha = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4 \times (\pi)^2 \times 0.1125}{(1.5)^2}$$

$$= 1.972\, Nm/rad$$

Hence, 1.972 Nm/rad is torsional spring constant of wire.

14.24 A body describes simple harmonic motion with an amplitude of 5cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm (b) 3 cm (c) 0 cm.

Answer:

Amplitude A = 5 cm = 0.05 m

Time period T = 0.2 s

(a) 5 cm

For displacement, x = 5 cm = 0.05 m

Acceleration $a = -\omega^2 x$

$$= -\left(\frac{2\pi}{T}\right)^2 x$$

$$= -\left(\frac{2\pi}{0.2}\right)^2 \times 0.05$$

$$= -5\pi^2\, m/s^2$$

Velocity $v = \omega\sqrt{A^2 - x^2} = \frac{2\pi}{T}\sqrt{(0.05)^2 - (0.05)^2} = 0$

(b) 3 cm

For displacement, x =3 cm = 0.03 m

Acceleration $a = -\omega^2 x$

$$= -\left(\frac{2\pi}{T}\right)^2 x$$

$$= -\left(\frac{2\pi}{0.2}\right)^2 \times 0.03$$

$= -3\pi^2 \, m/s^2$

Velocity $v = \omega\sqrt{A^2 - x^2} = \frac{2\pi}{T}\sqrt{(0.05)^2 - (0.03)^2} = \frac{2\pi}{0.2} \times 0.04 = 0.4\,\pi\, m/s$

(c) 0 cm

For displacement, x =0

Acceleration $a = -\omega^2 x = 0$

Velocity $v = \omega\sqrt{A^2 - x^2} = \frac{2\pi}{T}\sqrt{(0.05)^2 - 0} = 0.5\pi \; m/s$

14.25 A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It ispulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_o. [Hint: Start with the equation $x = a \cos(\omega t + \theta)$ and note that the initial velocity is negative.]

Answer:

Displacement $x = a \cos(\omega t + \theta)$

Velocity $v = \frac{dx}{dt} = -A\omega \sin(\omega t + \theta)$

At t = 0, $x = x_0$

$A\cos\theta = x_0 \qquad ....(i)$

Now, $\frac{dx}{dt} = -v_0 = A\omega \sin\theta$

$A\sin\theta = \frac{v_0}{\omega} \qquad ...(ii)$

Squaring and adding equations (i) and (ii), we get:

$$A^2(\cos^2\theta + \sin^2\theta) = x_0^2 + \left(\frac{v_0^2}{\omega^2}\right)$$

$$\therefore A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

Hence, the amplitude of resulting oscillation is $\sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$.