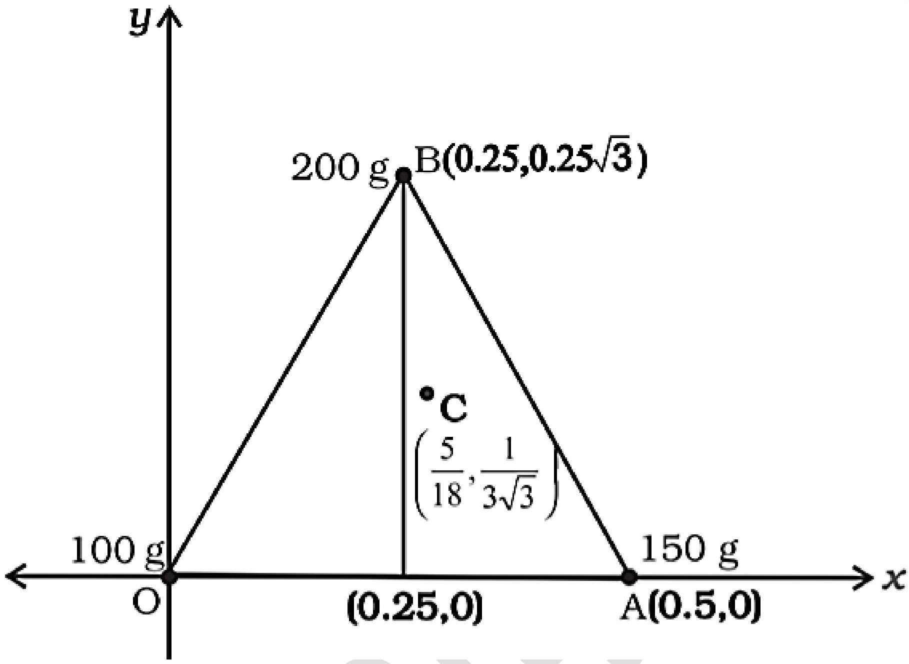
Chapter 7: SYSTEMS OF PARTICLES AND ROTATIONAL MOTION

EXAMPLES

1. Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are $100\text{g}, 150\text{g}, 200\text{g}$ **respectively. Each side of the equilateral triangle is** 0.5 m **long**

Answer:



Here, it is given that triangle ABO formed an equilateral triangle with points $(0,0), (0.5,0), (0.25,0.25\sqrt{3})$ respectively.

Now, let masses are $100\text{g}, 150\text{g}, 200\text{g}$ at O, A, B respectively.

So, $X = \dfrac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$

$$= \frac{100(0)+150(0.5)+200(0.25) \text{ g m}}{(100+150+200)\text{g}}$$

$$= \frac{75+50}{450}\text{m} = \frac{125}{450}\text{m} = \frac{5}{18}\text{m}$$

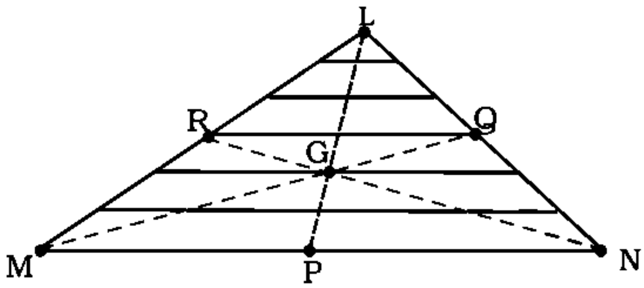
$$Y = \frac{100(0)+150(0)+200(0.25\sqrt{3})\text{g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450}\text{m} = \frac{\sqrt{3}}{9}\text{m} = \frac{1}{3\sqrt{3}}\text{m}$$

Hence, centre of Mass C is already shown in figure.

2. Find the centre of mass of a triangular lamina

Answer:



Here, lamina denotes as triangle LMN and it subdivided into narrow strips which is parallel to base MN.

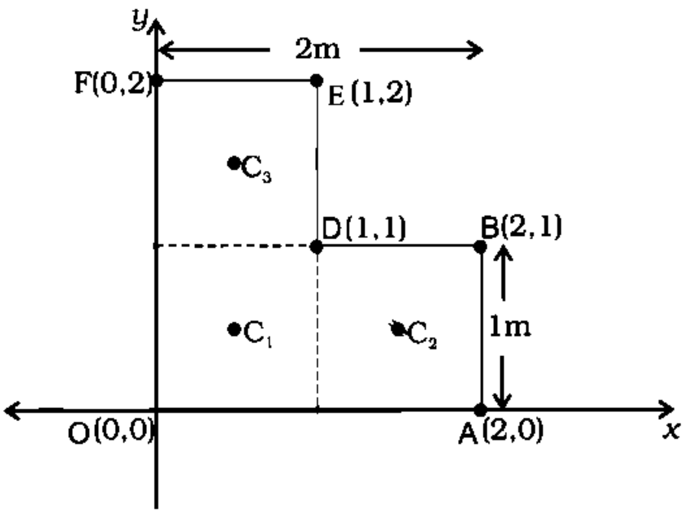
As a result of symmetry, each strip's centre of mass is at its midpoint. Now, if we join this mid-point, we get median LP. Centre of mass on the median LP.

Accordingly, it lies on median MQ and NR. This implies centre of mass lies on the point of concurrence medians that is centroid and it denoted as G in above figure.

3. Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimension as shown. The mass of the lamina is $3\ \text{kg}$

Answer: Let X and Y are axes. For L-shaped lamina we have coordinate as shown in figure. L-shaped lamina consists of three squares each of length one metre and mass is $1\ \text{kg}$, as lamina is uniform, according to question. So, Centre of mass C_1, C_2, C_3 of squares. So, their geometric centres have coordinates that is $(\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2})$ respectively.

Here, centre of L-shape is centre of mass of mass points.



Thus, $X = \dfrac{[1(\frac{1}{2})+1(\frac{3}{2})+1(\frac{1}{2})]\text{ kg m}}{(1+1+1)\text{kg}} = \dfrac{5}{6}\text{m}$

And $Y = \dfrac{[1(\frac{1}{2})+1(\frac{1}{2})+1(\frac{3}{2})]\text{ kg m}}{(1+1+1)\text{kg}} = \dfrac{5}{6}\text{m}$

Hence, centre of lies on OD as in above figure.

4. Find the scaler and vector products of two vectors a $= (3\hat{i}-4\hat{j}+5k)$ **, b** $= (-2\hat{i}+\hat{j}-3k)$

Answer: $a \times b = (3\hat{i}-4\hat{j}+5k)\times(-2\hat{i}+\hat{j}-3k)$

$= -6 - 4 - 15$

$= -25$

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{i}-\hat{j}-5k$$

$b \times a = -7\hat{i}+\hat{j}+5k$

5. Find the torque of a force $7\hat{i}+3\hat{j}-5k$ **about the origin. The force acts on a particle whose position vector is** $\hat{i}-\hat{j}+k$

Answer: It is given that $r = \hat{i}-\hat{j}+k$ and $F = 7\hat{i}+3\hat{j}-5k$

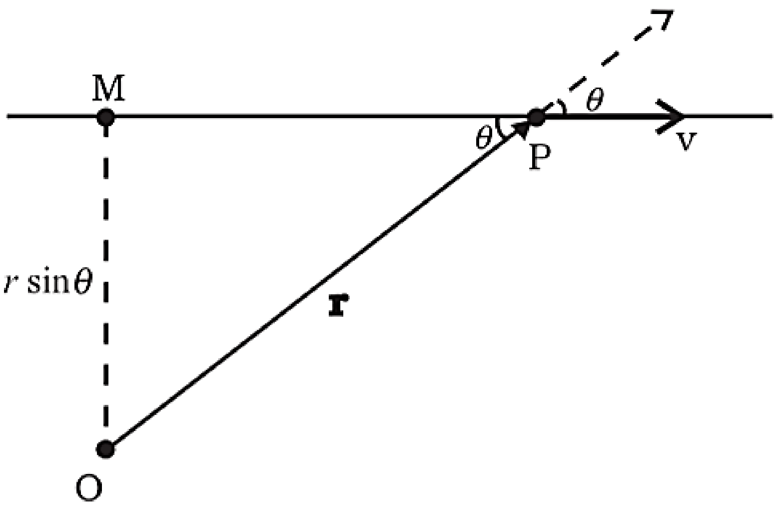
So, Torque $\tau = r \times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix}$

$= (5-3)\hat{i}-(-5-7)\hat{j}+(3-(-7))k$

$= 2\hat{i}+12\hat{j}+10k$

6. Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Answer: Here, let particle velocity v at P at some t. We have to calculate angular momentum of particle at O.



So, angular momentum is $l = r \times mv$

Magnitude $= mvr\sin\theta$ where θ is angle between v, r as in above figure.

Though, article change with time, line of direction of v remains same.

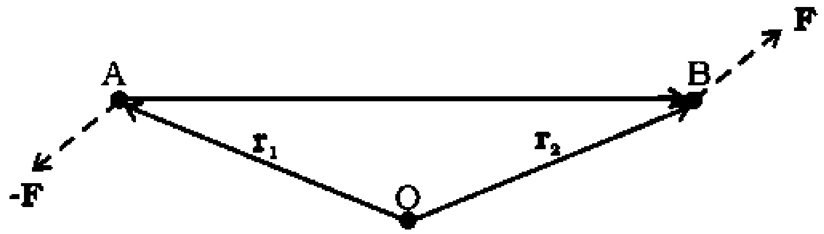
So, OM $= r\sin\theta$

Thus, direction of l is perpendicular to plane v, r and not change with time.

So, l remains same in magnitude and direction, so it conserved.

7. Show that moment of a couple does not depend on the point about which you take the moments

Answer:



According to figure, a couple acting on rigid body. Force F, $-F$ act at B and A respectively.

With respect to origin, these have position vector r_1, r_2 respectively.

Here, moment of couple $=$ sum of moments of two forces

$$= r_1 \times (-F) + r_2 \times F$$
$$= r_2 \times F - r_1 \times F$$
$$= (r_2 - r_1) \times F$$

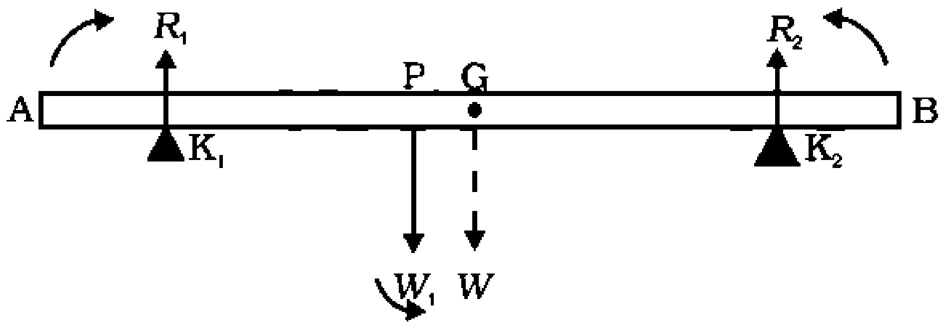
But $r_1 + AB = r_2$. Thus, $AB = r_2 - r_1$

Hence, moment of couple $= AB \times F$

Thus, it is independent of the origin.

8. A metal bar 70 cm long and 4.00 kg in mass supposed on two knife edges placed 10 cm from each end. A 6.00 kg load is suspended at 30 cm from one end. Find the reactions at the knife edges. (Assume the bar to be of uniform cross section and homogeneous)

Answer:



Above figure shows rod AB, positions of knife edges k_1, k_2, centre at G and suspended load at P.

Here weight W acts at centre of gravity at G.

Given that rod is uniform in cross section and homogeneous, thus G is at centre.

So, $AB = 70\text{ cm}$, $AG = 35\text{ cm}$, $AP = 30\text{ cm}$, $PG = 5\text{ cm}$, $AK_1 = AK_2 = 10\text{ cm}$,
$K_1G = K_2G = 25\text{ cm}$

Here, W $=$ weight of rod $=$ 4.00 kg and W_1 $=$ suspended load $= 6.00\text{ kg}$

And R_1, R_2 are normal reactions

Now, translation equilibrium of rod

$$R_1+R_2-W_1-W = 0 \qquad \text{(i)}$$

(W_1,W act vertically whereas R_1,R_2 act vertically up)

The moments of R_2, W_1 are anticlockwise whereas R_1 is clockwise.

So, for rotational equilibrium

$$-R_1(K_1G)+W_1(PG)+R_2(K_2G) = 0 \qquad \text{(ii)}$$

As we have W $= 4.00g$ N , W_1 $= 6.00\text{ g}$ where g $=$ acceleration

Let g $= 9.8\text{ m/s}^2$

So, from (i), we get

$$R_1+R_2-4.00g-6.00g = 0$$
$$R_1+R_2 = 10.00g\text{ N} \qquad \text{(iii)}$$
$$= 98.00\text{ N}$$

From(ii), we get:

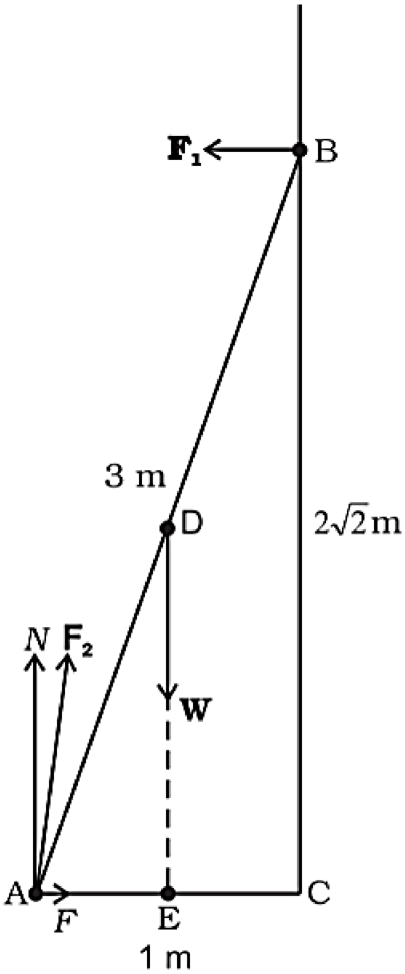
$$-0.25R_1+0.05W_1+0.25R_2 = 0$$
$$R_1-R_2 = 1.2g\text{ N} \qquad \text{(iv)}$$
$$= 11.76\text{ N}$$

From (iii) and (iv), we get:

$$R_1 = 54.88\text{ N} , R_2 = 43.12\text{ N}$$

9. A 3 m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in figure. Find the reaction forces of the wall and the floor

Answer:



Here AB $= 3$ m long at distance AC $= 1$ m from wall.

By using Pythagoras theorem, we get: BC $= 2\sqrt{2}$ m

Forces on ladder are its weight W and act at centre of gravity D and reaction force of wall and floor are respectively i.e., F_1 , F_2

Here, F_1 is perpendicular to wall and F_2 is in two components i.e. N for normal reaction and F for force of friction.

For transitional equilibrium (force in horizontal direction), $N\text{-}W = 0$ (i)

(force in vertical direction), $F\text{-}F_1 = 0$ (ii)

For rotational equilibrium, $2\sqrt{2}\ F_1\text{-}(\frac{1}{2})W = 0$ (iii)

$W = 20\ g = 20\times9.8\ N = 196.0\ N$

By (i), $N = 196.0\ N$

By (iii), $F_1 = \frac{W}{4\sqrt{2}} = \frac{196.0}{4\sqrt{2}} = 34.6$ N

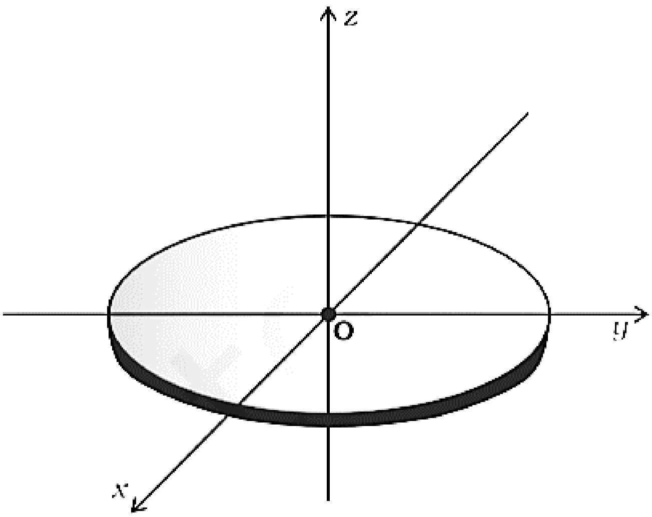
By (ii), $F = F_1 = 34.6$ N

$$F_2 = \sqrt{F^2 + N^2} = 199.0 \text{ N}$$

So, this force F_2 makes angle α, $\tan\alpha = \frac{N}{F} = 4\sqrt{2}, \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^\circ$

10. What is the moment of inertia of a disc about one of its diameters

Answer:



Let moment of inertia of is perpendicular on it around axis and centre is $\frac{MR^2}{2}$ where M denotes mass and R denotes radius.

Let disc be a planar body. So, let three concurrent axes from centre as x,y,z axes

Here x,y axes lie on plane and z axis is perpendicular on it

So, by using theorem of perpendicular axes, we get $I_z = I_x + I_y$

Now, moment of inertia is same as diameter

Thus, $I_x = I_y$

$$I_z = 2 I_x$$

$$I_z = \frac{MR^2}{2}$$

$$I_x = \frac{I_z}{2} = \frac{MR^2}{4}$$

So, moment of inertia about disc is $\frac{MR^2}{4}$

11. What is the moment of inertia of a rod of mass M , length l about an axis perpendicular to it through one end

Answer: We have given that mass M , length l where $I = \frac{Ml^2}{12}$

Now, use parallel axis theorem, we have $I' = I+Ma^2, a = \frac{1}{2}$

Then we get: $I' = M\frac{l^2}{12}+M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$

As l is half of the moment of inertia of mass $2M$ and length $2l$ from its midpoint,

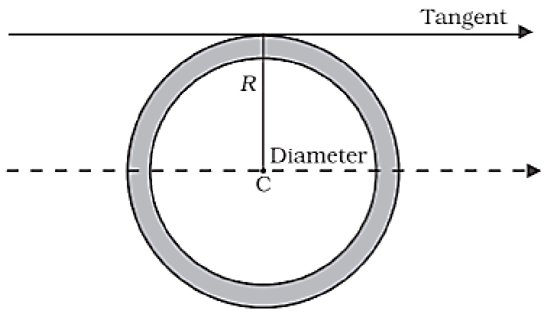
$I' = 2M\times\frac{4l^2}{12}\times\frac{1}{2} = \frac{Ml^2}{3}$

12. What is the moment of inertia of a ring about a tangent to the circle of the ring

Answer: Here, tangent of ring is parallel to on diameter of ring.

So, distance between two parallel axes $= R$ where R is radius

By using parallel axes theorem, we get:



$I_{tangent} = I_{dia} + MR^2$

$= \frac{MR^2}{2} + MR^2$

$= \frac{3}{2}MR^2$

13. Obtain $\omega = \alpha t+\omega_0$ from first principles

Answer: Here, angular acceleration is uniform

So, $\frac{d\omega}{dt} = \alpha =$ constant (i)

Now, integrate above equation, we get:

$\omega = \int \alpha\, dt + c$

$= \alpha t + c$ where α is constant

As we have given that $t = 0, \omega = \omega_0$

Thus from (i), we get:

At $t = 0, \omega = c = \omega_0$

Hence, $\omega = \alpha t + \omega_0$

14. The angular speed of a motor wheel is increased from 1200 rpm **to** 3120 rpm **in** 16 seconds **(i) What is its angular acceleration, assuming the acceleration to be uniform?**

(ii) How many revolutions does the engine make during this time?

Answer: (i) As we know $\omega = \omega_0 + \alpha t$

So, $\omega_0 =$ initial angular speed in rad/s

$$= 2\pi \times \text{angular speed in rev/s}$$
$$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$$
$$= \frac{2\pi \times 1200}{60} \text{ rad/s}$$
$$= 40\,\pi \text{ rad/s}$$

Accordingly, $\omega =$ final angular speed in rad/s

$$= \frac{2\pi \times 3120}{60} \text{ rad/s}$$
$$= 2\pi \times 52 \text{ rad/s}$$
$$= 104\,\pi \text{ rad/s}$$

So, angular acceleration $\alpha = \frac{\omega - \omega_0}{t} = 4\,\pi \text{ rad/s}^2$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$$= \left(40\pi\times16+\frac{1}{2}\times4\pi\times16^2\right)\text{rad}$$

$= (640\pi+512\pi)$ rad

$= 1152\ \pi$ rad

Hence, no. of revolutions $= \dfrac{1152\ \pi}{2\pi} = 576$

15. A cord of negligible mass is wound round the rim of a fly wheel of mass $20\ \text{kg}$ and radius $20\ \text{cm}$. A steady pull of $25\ \text{N}$ is applied on the cord as shown in Figure. The flywheel is mounted on a horizontal axle with frictionless bearings.

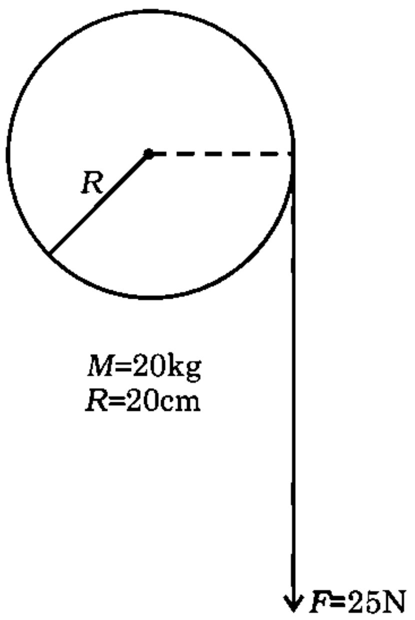
(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when $2\ \text{m}$ of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

Answer:



(a) As we know $I\alpha = \tau$

So, the torque $\tau = FR$

$= 25\times0.20$ Nm $\quad (\because\ R = 0.20\text{ m})$

$= 5.0$ Nm

Where I denotes moment of inertia of flywheel about axis i.e., $\dfrac{MR^2}{2}$

$$= \frac{20.0\times(0.2)^2}{2} = 0.4\ \text{kg m}^2$$

And α denotes angular acceleration i.e.

$$= \frac{5.0\ \text{N m}}{0.4\ \text{kg m}^2} = 12.5\ \text{s}^{-2}$$

(b) Here we have to find work done by the pull unwinding $2\ \text{m}$ of cord $= 25\ \text{N}\times 2\ \text{m} = 50\ \text{J}$

(c) Now, in this let ω $=$ final angular velocity

So, kinetic energy gained $= \dfrac{1}{2}\mathrm{I}\,\omega^2$

As wheel start from rest. Thus, $\omega^2 = \omega_0^2 + 2\alpha\theta,\ \omega_0 = 0$

Here, θ which is the angular displacement denotes the length of unwound string

$$= \frac{2\ \text{m}}{0.2\ \text{m}} = 10\ \text{rad}$$

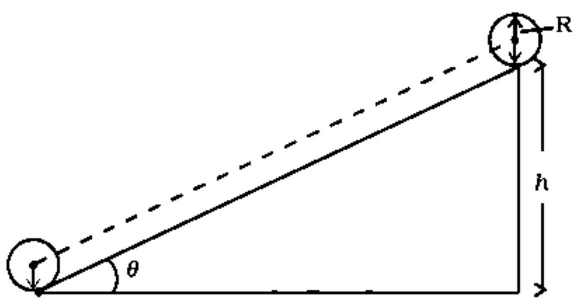
$$\omega^2 = 2\times 12.5\times 10.0 = 250(\text{rad/s})^2$$

Hence K.E. which is gained is $\dfrac{1}{2}\times 0.4\times 250 = 50\ \text{J}$

(d) As kinetic energy which is gained by wheel is equal to work which is done by force. So, due to friction, there is no loss of energy. Hence, answers are same.

16. Three bodies, a ring, a solid cylinder, and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

Answer:



Let there is no loss of energy of rolling body due to friction i.e., conservation of energy of rolling body. So, potential energy which lost by body in rolling down is equal to kinetic energy which is gained. As bodies start from rest. Thus, kinetic energy $=$ final kinetic energy.

As we know that $K = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$, where $v =$ final velocity

$$mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$v^2 = \left(\frac{2gh}{1+k^2/R^2}\right)$$

which is independent of mass of rolling body

So, for ring, $k^2 = R^2$

$$v_{ring} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$$

For cylinder $k^2 = \frac{R^2}{2}$

$$v_{disc} = \sqrt{\frac{2gh}{1+1/2}} = \sqrt{\frac{4gh}{3}}$$

For sphere $k^2 = \frac{2R^2}{5}$

$$v_{sphere} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10gh}{7}}$$

EXERCISE

Question 7. 1: Give the location of the centre of mass of a (i) sphere, (ii-cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

Answer: we have given that the four cases, as the mass is uniform, centre of mass is exist at their respective geometrical centres.

No, it is not required that the mass of a body should be lie on the body.

For example, in case of a circular ring, centre of mass will be lie at the centre of the ring, where there is no mass.

Question 7.2: In the HCl molecule, the separation between the nuclei of

the two atoms is about $1.27\,\text{A}\left(1\text{A}=10^{-10}\,\text{m}\right)$**. Find the approximate location of the CM of the molecule, given that a chlorine atom is about**

35.5 **times as massive as a hydrogen atom and nearly all the mass of an**

atom is concentrated in its nucleus.

Answer: Given:

Distance between atom H and Cl $= 1.27\text{A}$

Mass of H atom $= m$

Mass of Cl atom $= 35.5\text{m}$

Let us suppose that centre of mass of the system lie at a distance x from the Cl atom.

And also distance of the centre of mass from the H atom $=(1.27-x)$

Let us suppose that the centre of mass of the molecule lies at the origin.

Therefore, we can write as

$$\frac{\text{m}(1.27 - \text{x}) + 35.5\text{mx}}{\text{m} + 35.5\text{m}} = 0$$

$$\text{m}(1.27 - \text{x}) + 35.5\text{mx} = 0$$

$$1.27 - \text{x} = - 35.5\text{x}$$

$$\text{x} = \frac{- 1.27}{(35.5\text{-}1)} = - 0.037\ \text{A}$$

Here, the negative sign show that the centre of mass lies at the left-hand side of the molecule.

So, centre of mass of HCl molecule exist $0.037\ \text{A}$ from the Cl atom.

Question 7.3: A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Answer: No change

The child moving arbitrary on a trolley with velocity v. So, the running of the child will produce no effect on the velocity of the centre of mass of the trolley. Because the force due to the boy's motion is Internal forces

Produce no effect on the motion of the bodies on which they lie. Since no external force is include in the boy trolley system, the boy’s motion will induce no change in the velocity of the centre of mass of the trolley.

Question 7.4: Show that the area of the triangle contained between the vectors a and b is one half of the magnitude of $a \times b$.

Answer: let us suppose two vectors $\overrightarrow{OK} = |\vec{A}|$ and $\overrightarrow{OM} = |\vec{b}|$ $\overrightarrow{OM} = |\vec{b}|$ inclined with angle θ.

In Δ OMN,:

$$\sin\theta = \frac{MN}{OM} = \frac{MN}{|\vec{b}|}$$

$MN = |\bar{b}| \sin\theta$

$|\vec{a} \times \vec{a}| = |\vec{a}|\ \vec{b}\ \sin\theta$

$= OK.MN \times \frac{2}{2}$

$= 2\times$ Area of Δ OMK

Area of $\triangle OMK = \frac{1}{2}|\vec{a} \times \vec{b}|$

Question 7.5: Show that $a \,.\, (b \times c)$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors, a, b and c

Answer: let us take the origin O of A parallelepiped and sides a, b, and c.

Volume of the parallelepiped $= abc$

$\overline{OC} = \vec{a}$

$\overline{OB} = \vec{b}$

$\overline{OC} = \vec{c}$

Let us suppose $\vec{n}$ be a unit vector which are perpendicular to b and c.

so, $\vec{n}$ and a have in the same direction.

$\therefore\ \vec{b} \times \vec{c} = bc \sin\theta$

$= bc \sin 90^\circ \hat{n}$

$= bc\hat{n}$

$\bar{a} \times (\bar{b} \times \vec{c})$

$= a\,.\,(bc\hat{n})$

$= abc \cos\theta\ \hat{n}$

$= abc \cos 0^\circ$

$= abc$

= Volume of the parallelepiped

Question 7.6: Find the components along the x, y, z axes of the angular momentum I of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_x and p_x. Show that if the particle moves only in the $x-y$ plane the angular momentum has only a z-component.

Answer: $l_x = yp_z - zp_y l_y$

$= zp_x - xp_z l_z$

$= xp_y - yp_x$

let the Linear momentum of the particle, $\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

Position vector of the particle, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Angular momentum, $\vec{l} = \vec{r} \times \vec{p}$

$= (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\mathbf{k}) \times \left(p_{\mathbf{k}}\hat{\mathbf{i}} + p_{\mathbf{j}}\hat{\mathbf{j}} + p_{\mathbf{n}}\mathbf{k}\right)$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \mathbf{k} \\ x & y & z \\ p_{\mathrm{i}} & p_3 & p_2 \end{vmatrix}$$

$$l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}} + l_z\mathbf{k} = \hat{\mathbf{i}}\left(yp_z - zp_y\right) - \hat{\mathbf{j}}\left(xp_z - zp_x\right) + \mathbf{k}\left(xp_y - zp_x\right)$$

Comparing the similar coefficients of $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and k, then we get:

$$\left.\begin{aligned} 1_x &= yp_z - zp_y \\ 1_y &= xp_z - zp_x \\ 1_z &= xp_y - yp_x \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{i})$$

The particle rotates in the $x-y$ plane.

Hence, the z-component of the position vector and linear momentum vector are given zero, i.e., $z = p_z = 0$

Thus, equation (i) decompes to:

$$\left.\begin{aligned} l_x &= 0 \\ t_y &= 0 \\ t_z &= xp_y - yp_x \end{aligned}\right\}$$

Therefore, when the particle is move in the x-y plane, so angular momentum direction is along the z-direction.

Question 7. 7: Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two-particle system is the same whatever be the point about which the angular momentum is taken.

Answer: Let at fixed distance two particles be at points P and Q.

Angular momentum along point P:

$$\vec{L}_P = mv \times 0 + mv \times dn \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
$$= mvd$$

Angular momentum along point Q:

$$\vec{L}_Q = mv \times d + mv \times 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$
$$= mvd$$

Let us take a point R, whose distance is y from point Q, i.e.,

$$QR = y$$

$$\therefore PR = d - y$$

Angular momentum along point R:

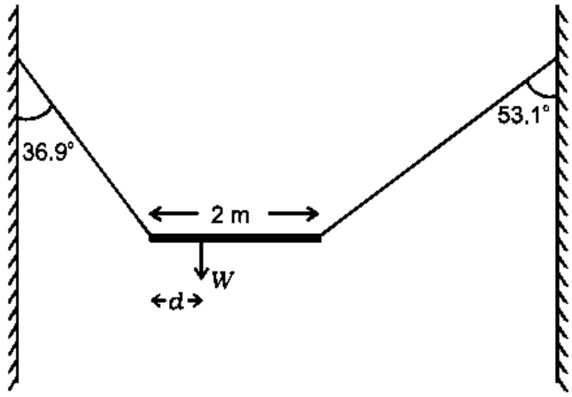
$$\vec{L}_R = mv \times (d - y) + mv \times y$$
$$= mvd - mvy + mvy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$
$$= mvd$$

Compare (i),(ii), and (iii), we get:

$$\vec{L}_P = \vec{L}_Q = \vec{L}_R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

We can conclude from equation (iv) that the angular momentum of a system does not depend on the point about which it is taken.

Question 7.8: A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in the given Fig. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is $2\,\text{m}$ long. Calculate the distance d of the centre of gravity of the bar from its left end.



Answer:

Length of the bar, $l = 2\text{m}$

T_1 and T_2 are the tensions of the body produced left and right strings respectively

At translation equilibrium,

$$T_1 \sin 36.9^\circ = T_2 \sin 53.1$$

$$\frac{T_1}{T_2} = \frac{\sin 53.1^\circ}{\sin 36.9}$$

$$= \frac{0.800}{0.600} = \frac{4}{3}$$

$$\Rightarrow T_1 = \frac{4}{3} T_2$$

For rotational equilibrium, let us take the torque along the centre of gravity, we have:

$$T_1 \cos 36.9 \times d = T_2 \cos 53.1(2 - d)$$

$$T_1 \times 0.800d = T_2 0.600(2 - d)$$

$$\frac{4}{3} \times T_2 \times 0.800d = T_2[0.600 \times 2 - 0.600d]$$

$$1.067d + 0.6d = 1.2$$

$$= 0.72 \text{ m}$$

Hence, the C. G (centre of gravity) of the free body bar lies $= 0.72 \text{ m}$ from its left end hand side of the string.

Question 7.9: A car weighs $1800\,\text{kg}$ The distance between its front and back axles is 1.8m. Its centre of gravity is 1.05m behind the front axle. Determine the force exerted by the level ground on each front wheel and each hack wheel

Answer: Mass of the car, $m = 1800\,\text{kg}$

Distance between the back axle front axle, $d = 1.8\text{m}$

Distance between the C.G. (centre of gravity) and the back axle $=1.05\mathrm{m}$

R_f and R_b are the forces generated by the level ground on the front and back wheels respectively

For translational equilibrium:

$$R_f + R_b = mg$$
$$= 1800 \times 9.8$$
$$= 17640 \text{ N} \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

For rotational equilibrium, when we take the torque about the C.G., we have:

$$R_f(1.05) = R_b(1.8 - 1.05)$$
$$R_r \times 1.05 = R_5 \times 0.75$$
$$\frac{R_T}{R_b} = \frac{0.75}{1.05} = \frac{5}{7}$$
$$\frac{R_b}{R_f} = \frac{7}{5}$$

$$R_b = 1.4R_f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

from (i) and (ii), we get:

$$1.4R_f + R_f = 17640$$
$$R_f = \frac{17640}{2.4} = 7350 \text{ N}$$
$$\therefore R_b = 17640 - 7350 = 10290 \text{ N}$$

Therefore, the force pull by each front wheel $\frac{7350}{2} = 3675\,\text{N}$

The force pull by each back wheel $= \frac{10290}{2} = 5145\,\text{N}$

Question 7.10: Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2MR^2/5$ where M is the mass of the sphere and R is the radius of the sphere.

Given the moment of inertia of a dise of mass M and radius R about any of its diameters to be $2MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

Answer: $\frac{7}{5}MR^2$

Moment of inertia (M.1.) of a sphere along its diameter $\frac{2}{5}MR^2$

$\text{M.I.} = \frac{3}{5}\text{MR}^2$

As According to the theorem of parallel axes, the moment of inertia of a body along any axis which is equal to the product of its mass sum of moment of inertia of the body about a parallel axis passing through its centre of mass and the square of the distance between the two parallel axes.

The M.I. along a tangent of the sphere $= \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$

(b) $\frac{3}{2}MR^2$

The moment of inertia of a disc along its diameter $= \frac{1}{4}MR^2$

As by the theorem of perpendicular axis, the moment of inertia of a plane body (lamina) along an axis perpendicular to plane which is equal to the sum of the moments of inertia along two perpendicular axes similar with perpendicular axis and exist in the plane of the body.

The M.I. of the disc along its centre $= \frac{1}{4}MR^2 + \frac{1}{4}MR^2 = \frac{1}{2}MR^2$

Using the theorem of parallel axes.

When moment of inertia along an axis normal to the disc and passing through a point on its edge

$= \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$

Question 7.11: Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Answer: Let us assume m and r be the masses and radius of the hollow cylinder and the solid sphere.

The moment of inertia of the hollow cylinder along its standard axis, $I_1 = mr^2$

When an axis passing through its centre the moment of inertia of the solid sphere about $I_{11} = \frac{2}{5}mr^2$

We know the relation:

$\tau = I\alpha$

Where, $\alpha =$ Angular

Acceleration $\tau =$ Torque

I $=$ Moment of inertia

For the hollow cylinder, $\tau_1 = l_1\alpha_1$

For the solid sphere, $\tau_{11} = I_{11}\alpha_{11}$

When an equal torque is applied to both the bodies, $\tau_1 = \tau_2$

$$\frac{a_{11}}{a_1} = \frac{I_1}{I_0} = \frac{mr^2}{\frac{2}{5}mr^2} = \frac{2}{5}m^2$$

$\alpha_t > \alpha_1$ ....(i)

Now, we know the relation:

$\theta = \omega_0 + at$

Where,

$\omega_0 =$ Initial angular velocity

$t =$ Time of rotation $\omega =$

For final angular velocity

For equal ω_0 and t, we have

$\omega \propto a$ ....(ii)

By (i) and (ii), we can write:

$\omega_{11} > \omega_1$

So, Angular velocity of the solid sphere will be greater than that Angular velocity of the hollow cylinder

Question 7.12: A solid cylinder of mass $20\,\text{kg}$ rotates about its axis with angular speed 100rads^{-1} The radius of the cylinder is 0.25m .What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Answer

Mass of the cylinder, $m = 20\,\text{kg}$

Angular speed, $\omega = 100\text{rads}^{-1}$

Radius of the cylinder, $r = 0.25\text{m}$

moment of inertia for the solid cylinder.

$$I = \frac{mr^2}{2}$$

$$= \frac{1}{2} \times 20 \times (0.25)^2$$
$$= 0.625\,\text{kg m}^2$$

So Kinetic energy $= \frac{1}{2} I\omega^2$

$$= \frac{1}{2} \times 6.25 \times (100)^2 = 3125J$$

Then, Angular momentum, $\text{L} = \text{l}\omega$

$$= 6.25 \times 100$$
$$= 62.5\text{Js}$$

Question 7.13: A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to $2/5$ times the initial value? Assume that the turntable rotates without friction. Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do your account for this increase in kinetic energy?

Answer: 100 rev/min

Initial angular velocity, $\omega_1 = 40\,\text{rev/min}$

Final angular velocity ω_2

The moment of inertia when boy stretched his hands $= I_1$

The moment of inertia when boy fold his hands $= I_2$

The two moments of inertia are related as:

$$l_2 = \frac{2}{5} I_1$$

Since no external force will be acts on the boy, the angular momentum L is a uniform

Hence, for the two conditions, we can write:

$$\text{t}_2\omega_2 = \text{l}_1\omega_1$$

$$\omega_2 = \frac{\text{l}_1}{\text{l}_2}\omega_1$$

$$= \frac{\text{I}_1}{\frac{2}{5}\text{I}_1} \times 40 = \frac{5}{2} \times 40$$

$$= 100 \text{ rev / min}$$

(b)Final $K.E. = 2.5$ Initial $K.E.$

Final kinetic rotation, $E_F = \frac{1}{2} I_2 \omega_1^2$

Initial kinetic rotation, $E_1 = \frac{1}{2} I_1 \omega_1^2$

$$\frac{E_F}{E_I} = \frac{\frac{1}{2} I_2 \omega_2^2}{\frac{1}{2} I_1 \omega_1^2}$$

$$= \frac{2}{5} \frac{I_1}{I_1} \frac{(100)^2}{(40)^2}$$

$$= \frac{2}{5} \times \frac{100 \times 100}{40 \times 40}$$

$$= \frac{5}{2} = 2.5$$

$$\therefore E_F = 2.5 E_1$$

The increase in the rotational kinetic energy is effect on the internal energy of the boy.

Question 7.14: A rope of negligible mass is wound round a hollow cylinder of mass $3\,\text{kg}$ and radius $40\,\text{cm}$. What is the angular acceleration of the cylinder if the rope is pulled with a force of $30\,\text{N}$? What is the linear acceleration of the rope? Assume that there is no slipping.

Answer

Mass of the hollow cylinder, $3\,\text{kg}$

Radius of the hollow cylinder, $r = 40\,\text{cm} = 0.4\,\text{m}$

Applied force, $F = 30\,\text{N}$

Moment of inertia of the hollow cylinder along the geometric axis.

$$I = mr^2$$

$$= 3 \times (0.4)^2 = 0.48\,\text{kg m}^2$$

Torque, $\tau = F \times r$

$$= 30 \times 0.4 - 12\,\text{Nm}$$

For angular acceleration α,

torque can be found by the relation:

$r = l\alpha$

$\alpha = \frac{r}{l} = \frac{12}{0.48}$

$= 25 \text{rad s}^{-2}$

Linear acceleration $= r\alpha = 0.4 \times 25 = 10 \text{m s}^{-2}$

Question 7.15: To maintain a rotor at a uniform angular speed of $200 \text{rad } s^{-1}$, an engine needs to transmit a torque of 180Nm. What is the power required by the engine?

(Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

Answer

Angular speed of the rotor, $6 = 200 \text{rad/s}$

Torque needed for this, $\tau = 180 \text{Nm}$

The power of the rotor (P) can by the torque and angular speed by the relation:

$P = \tau\omega$

$= 180 \times 200 = 36 \times 10^3$

$= 36$ kW

Hence, the power needed for the engine is 36 kW.

Question 7.16: From a uniform disk of radius R, a circular hole of radius $R/2$ is cut out. The centre of the hole is at $R/2$ from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

Answer: $R/6$, is the radius of the original centre of the body and opposite of the centre of when cut the portion at the centre

Mass per unit area of the original disc $= \sigma$

Radius of the original disc $= R$

Mass of the original disc, $M = \pi R^2 \sigma$

When the disc cut from the centre portion:

Radius of the smaller disc $= \frac{R}{2}$

Mass of the smaller disc, $M = \pi \left(\frac{R}{2}\right)^2 \sigma = \frac{1}{4}\pi R^2 \sigma = \frac{M}{4}$

Let us assume O and O’ be the centres of the original disc and the disc removed from the original. We know the definition of the centre of mass, the centre of mass of the original disc is assumed to be concentrated at O, while that of the smaller disc is assumed to be concentrated at O’.

It is given that:

$$OO' = \frac{R}{2}$$

After the smaller disc has been cut from the original disc, the remained part is taken as to be a system of two masses. The two masses are:

M(concentrated at O), and

$-MP\left(=\frac{M}{4}\right)$ concentrated at O’

(The negative sign show that this portion has been cut from the original disc.)

Let x be the distance from which the centre of mass of the remaining part when it is shifts from point O.

We know the centres of masses of two masses is as by this relation:

$$x = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

Of the given system,

we can write distance as by this:

$$x = \frac{M \times 0 - M \times \left(\frac{R}{2}\right)}{M + \left(-M'\right)} = \frac{\frac{-M}{4} \times \frac{R}{2}}{M - \frac{M}{4}}$$

$$= \frac{-MR}{8} \times \frac{4}{3M} = \frac{-R}{6}$$

(The negative sign show that the centre of mass gets shifted toward the opposite of point O.)

Question 7.17: A metre stick is balanced on a knife edge at its centre. When two coins, each of mass $5\,\text{g}$ are put one on top of the other at the $12.0\,\text{cm}$ mark, the stick is found to be balanced at $45.0\,\text{cm}$.What is the mass of the metre stick?

Answer

Let us assume W and W’ be the weights of the metre stick and the coin.

The mass of the metre stick is consider as at its mid-point, i.e., at the $50\,\text{cm}$ mark.

Mass of the meter stick $= m$

Mass of each coin, $m = 5\text{g}$

When the coins are taken 12cm away from the end P, the centre of mass gets displace by 5cm from point R toward the end P. The centre of mass is exist at a distance of 45cm from point P.

Total torque will be conserved for rotational equilibrium along point R.

$$10 \times g(45-12) - m' g(50-45) = 0$$

$$\therefore m' = \frac{10 \times 33}{5} = 66\text{g}$$

So, the mass of the metre stick $= 66\text{g}$.

Question 7.18: A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?

Answer:

(a) Yes

(b) Yes

(c) for the smaller inclination

(a) sphere's mass $= m$

Plane's height $= h$

Sphere's velocity at the bottom of the plane $= v$

At the top of the plane, Sphere's total energy $=$ Potential energy $= mgh$

At the bottom of the plane, sphere both energies will be considered translational and rotational kinetic energies.

Hence, total energy $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

On using the law of conservation of energy, we can write:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh \text{ .... (i)}$$

For a solid sphere, the moment of inertia along its centre, $I = \frac{2}{5}mr^2$

Hence, equation (i) becomes:

$$\frac{1}{2}\text{mv}^2 + \frac{1}{2}\left(\frac{2}{5}\text{mr}^2\right)$$

$$\frac{1}{2}\text{v}^2 + \frac{1}{5}\text{r}^2\text{e}^2 = \text{gh}$$

As we know, $v = re$

$$\therefore \frac{1}{2}v^2 + \frac{1}{5}v^2 = gh$$

$$v^2\left(\frac{7}{10}\right) = gh$$

$$v = \sqrt{\frac{10}{7}gh}$$

Hence, the Sphere's velocity at the bottom depends on height (h) and acceleration due to gravity (g). Both height and acceleration are constants. Therefore, the Sphere's velocity at the bottom remains constant from whenever inclined plane's sphere is rolled.

(b), (c) let us assume two inclined planes with angle θ_1 and θ_2 related as:

The acceleration induced in the sphere when it rolls down the plane inclined at angle θ_1 is: $g\sin\theta_1$

Different type of forces acting on the sphere are:

Normal reaction to the sphere is R_1

Similarly, the acceleration induced in the sphere when it rolls down the plane inclined at θ_2 is : $g \sin \theta_2$

The different forces acting on the sphere are:

R_2 is the normal reaction to the sphere.

$\theta_2 > \theta_1$;

$\sin\theta_2 > \sin\theta_1$ ………………………(i)

$a_2 > a_1$……………………..(ii)

Initial velocity, $u = 0$

Final velocity, $v =$ Constant

By Using the first equation of motion,

$v = u + at$ ……………………………(iii)

Using equations (ii) and (iii), we get:

$t_2 < t$

Hence, the sphere have been taken longer time at reach the bottom of the inclined as compare to plane having the smaller inclination.

Question 7.19: A hoop of radius $2\,\text{m}$ *weighs* $100\,\text{kg}$. It rolls along a horizontal floor so that its centre of mass has a speed of $20\,\text{cm/s}$. How much work has to be done to stop it?

Answer: hoop's mass, $r = 2\,\text{m}$

Hoop's mass, $m = 100\,\text{kg}$

Hoop's Velocity, $v = 20\,\text{cm/s} = 0.2\,\text{m/s}$

Total energy of the hoop $=$ Translational K.E. $+$ Rotational K.E.

$$E_T = \frac{1}{2}mv^2 + \frac{1}{2}lv^2$$

Moment of inertia of the hoop along its centre, $I = mr^2$

$$E_1 = \frac{1}{2}mv^2 + \frac{1}{2}\left(mr^2\right)\omega^2$$

We know that $v = r\omega$

$$\therefore \mathrm{E}_1 = \frac{1}{2}\mathrm{mv}^2 + \frac{1}{2}\mathrm{mr}^2\omega^2$$

$$= \frac{1}{2}\mathrm{mv}^2 + \frac{1}{2}\mathrm{mv}^2 = \mathrm{mv}^2$$

Work need done for stopping the hoop is equal to the total energy of the hoop.

$$\therefore\ W = mv^2 = 100 \times (0.2)^2 = 4\,\text{J}$$

Question 7.20: The oxygen molecule has a mass of $5.30 \times 10^{-26}\,\text{kg}$ and a moment of inertia of $1.94 \times 10^{-46}\,\text{kg m}^2$ about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is $500\,\text{m/s}$ and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Answer : oxygen molecule's mass, $m = 5.30 \times 10^{-26}\,\text{kg}$

Moment of inertia, $I = 1.94 \times 10^{-46}\,\text{kg m}^2$

Oxygen molecule's velocity, $v = 500\,\text{m/s}$

Separation of two atoms of the oxygen molecule $= 2r$

Mass of each oxygen atom $= \dfrac{m}{2}$

Hence, moment of inertia I ,

$$\left(\frac{m}{2}\right)r^2 + \left(\frac{m}{2}\right)r^2 = mr^2$$

$$r = \sqrt{\frac{1}{m}}$$

$$\sqrt{\frac{1.94 \times 10^{-4}}{5.36 \times 10^{-26}}} = 0.60 \times 10^{-10}$$

We have given that:

$$KE_{rot} = \frac{2}{3}KE_{trans}$$

$$\frac{1}{2}I\omega^2 = \frac{2}{3} \times \frac{1}{2} \times mv^2$$

$$mr^2\omega^2 = \frac{2}{3}mv^2$$

$$\omega = \sqrt{\frac{2}{3}}\frac{v}{r}$$

$$= \sqrt{\frac{2}{3}} \times \frac{500}{0.6 \times 10^{-10}}$$

$$= 6.80 \times 10^{12} \text{rad/s}$$

Question 7.21: A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s How far will the cylinder go up the plane?

How long will it take to return to the bottom?

Answer: Given

Initial velocity, $v = 5 \text{ m/s}$

Angle of inclination, $\theta = 30^\circ$

Height reached by the cylinder $= h$ Energy required of the cylinder at point A:

$$KE_{rot} = KE_{trans}$$

$$\frac{1}{2}I\omega^2 = \frac{1}{2}mv^2$$

Energy of the cylinder on point $B = mgh$

By using the law of conservation of energy, we know that

$$\frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 = mgh$$

Moment of inertia of the solid cylinder,

$$I=\frac{1}{2}mr^2$$

$$\therefore \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2+\frac{1}{2}mv^2=mgh\frac{1}{4}mr^2c^2+\frac{1}{2}mv^2=mgh$$

But we have the relation, $v=re$

$$\therefore \frac{1}{4}v^2+\frac{1}{2}v^2=gh$$

$$\frac{3}{4}v^2=gh$$

$$\therefore h=\frac{3v^2}{4g}$$

$$=\frac{3}{4}\times\frac{5\times 5}{9.8}=1.91\text{m}$$

$In \triangle \text{ABC}:$

$$\sin\theta=\frac{BC}{AB}$$

$$\sin 30^\circ=\frac{h}{\text{AB}}$$

$$\text{AB}=\frac{1.91}{0.5}=3.82\,\text{m}$$

Hence, the cylinder will travel $3.82\,\text{m}$ up the inclination plane.

When K is the radius of gravitation , the velocity of the cylinder at the instance when it rolls back to the bottom is given by the relation:

$$v=\left(\frac{2gh}{1+\frac{K^2}{R^2}}\right)^{\frac{1}{2}}$$

$$\therefore v=\left(\frac{2\,\text{gAB}\sin\theta}{1+\frac{K^2}{R^2}}\right)^{\frac{1}{2}}$$

For the solid cylinder, $K^2=\frac{R^2}{2}$

$$\therefore v = \left(\frac{2gAB\sin\theta}{1+\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= \left(\frac{4}{3}\text{gAB}\sin\theta \right)^{\frac{1}{2}}$$

The time will taken to return to the bottom is:

$$\text{t} = \frac{\text{AB}}{\text{v}}$$

$$= \frac{\text{AB}}{\left(\frac{4}{3}\text{gAB sin }\theta \right)^{2}} = \left(\frac{3\text{AB}}{4\text{gsin }\theta} \right)^{2}$$

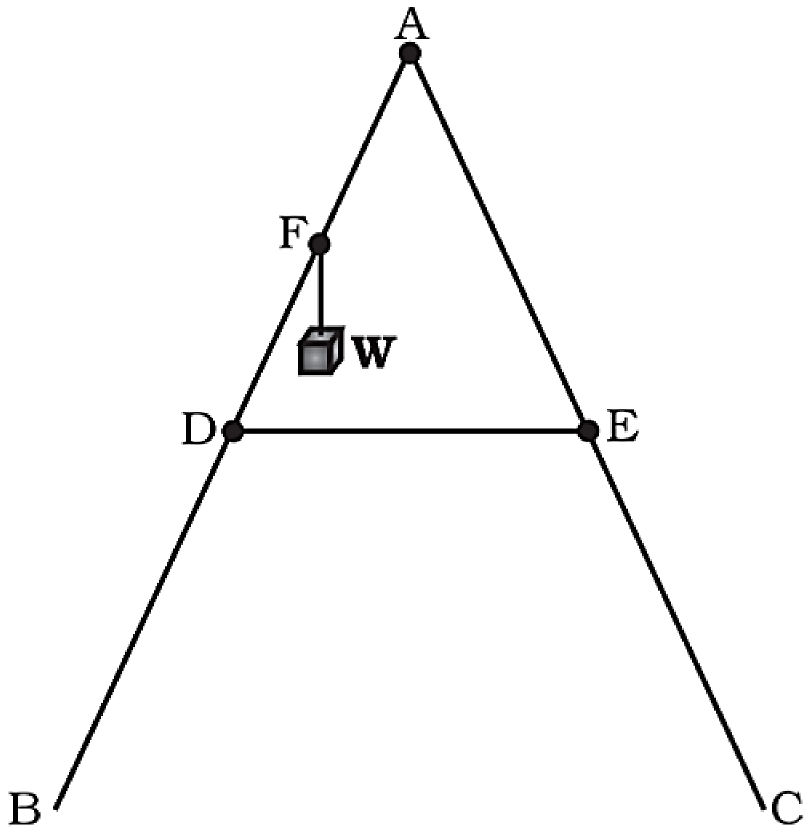
$$= \left(\frac{11.46}{19.6} \right)^{\frac{1}{2}} = 0.764 \text{ s}$$

Therefore, the total time taken by the cylinder to return from back to the bottom is $(2\times 0.764)1.53s$.

Additional Exercise

Question 7.22: As shown in Fig., the two sides of a step ladder BA and CA are $1.6\,\text{m}$ long and hinged at Λ. A rope D E, $0.5\,\text{m}$ is tied half way up. A weight $40\,\text{kg}$ is suspended from a point F, $1.2\,\text{m}$ from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (take $g = 9.8\,\text{m}/\text{s}^2$)

(Hint: Consider the equilibrium of each side of the ladder separately.)



Answer: Given

N_B = Force pull by the ladder on the floor point B

N_C = Force pull by the ladder on the floor point C

T = Tension in the rope

$\mathrm{BA} = \mathrm{CA} = 1.6\,\mathrm{m}$
$\mathrm{DE} = 0.5\,\mathrm{m}$
$\mathrm{BF} = 1.2\,\mathrm{m}$

Mass of the weight, $m = 40\,\mathrm{kg}$

Draw a perpendicular from A on the BC. It cut DE at mid-point H.

$\triangle\mathrm{ABI}$ *and* $\triangle\mathrm{AIC}$ are similar

$\therefore \quad \mathrm{BI} = \mathrm{IC}$

Hence, I is the mid-point of BC.

$DE \parallel BC$

$\mathrm{BC} = 2 \times \mathrm{DE} = 1\mathrm{m}$

$\mathrm{AF} = \mathrm{BA} - \mathrm{BF} = 0.4\,\mathrm{m}$ ....(i)

D is the mid-point of AB.

Hence, we can write:

$\mathrm{AD} = \frac{1}{2} \times \mathrm{BA} = 0.8\,\mathrm{m}$ ....(ii)

From (1) and (2), we get:

$\mathrm{FE} = 0.4\,\mathrm{m}$

Hence, F is the mid-point of AD.

$\mathrm{FG} \parallel \mathrm{DH}$ and F is the mid-point of AD. Then we can say that, G will be the mid-point of AH.

$\triangle \mathrm{AFG}$ and $\triangle \mathrm{ADH}$ are similar

$$\therefore \frac{\mathrm{FG}}{\mathrm{DH}} = \frac{\mathrm{AF}}{\mathrm{AD}}$$

$$\frac{\mathrm{FG}}{\mathrm{DH}} = \frac{0.4}{0.8} = \frac{1}{2}$$

$$\mathrm{FG} = \frac{1}{2}\mathrm{DH}$$

$$= \frac{1}{2} \times 0.25 = 0.125 \text{ m}$$

In $\triangle \mathrm{ADH}$

$$\mathrm{AH} = \sqrt{\mathrm{AD}^2 - \mathrm{DH}^2}$$

$$= \sqrt{(0.8)^2 - (0.25)^2} = 0.76 \text{ m}$$

In the case translational equilibrium of the ladder, the downward force upward force should be equal to the upward force.

$N_c + N_B = mg = 392$ ....(iii)

For rotational equilibrium of the ladder, the total momentum about A is:

$$- \mathrm{N}_B \times \mathrm{BI} + \mathrm{m\,g} \times \mathrm{FG} + \mathrm{N}_\mathrm{C} \times \mathrm{Cl} + \mathrm{T} \times \mathrm{AG} - \mathrm{T} \times \mathrm{AG} = 0$$

$$- \mathrm{N}_\mathrm{B} \times 0.5 + 40 \times 9.8 \times 0.125 + \mathrm{N}_\mathrm{C} \times (0.5) = 0$$

$$\left(\mathrm{N}_\mathrm{C} - \mathrm{N}_\mathrm{B}\right) \times 0.5 = 49$$

$\mathrm{N}_\mathrm{C} - \mathrm{N}_\mathrm{B} = 98$ ....(4)

Adding equation (3) and (4) we get:

$$\mathrm{N}_\mathrm{C} = 245\mathrm{N}$$

$$\mathrm{N}_B = 147\mathrm{N}$$

For rotational equilibrium of the side AB, let us assume the moment of A.

$- N_B \times BI + mg \times FG + T \times AG = 0$

$- 245 \times 0.5 + 40 + 9.8 \times 0.125 + T \times 0.76 = 0$

$0.76T = 122.5 - 49$

$\therefore T = 96.7N$

Question 7.23: A man stands on a rotating platform, with his arms stretched horizontally holding a $5\,kg$ weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from $90\,cm$ to $20\,cm$. The moment of inertia of the man together with the platform may be taken to be constant and equal to $7.6\,kg\,m^2$. What is his new angular speed? (Neglect friction.)

Is kinetic energy conserved in the process? If not, from where does the change come about?

Answer: (a) $58.88 rev/min$ **(b)** No

(a) Moment of inertia of the man $= 7.6\,kg\,m^2$

Moment of inertia stretched hands to a distance of $90\,cm$

$2 \times mr^2$

$= 2 \times 5 \times (0.9)^2$

$= 8.1\ kg\ m^2$

Initial moment of inertia of the system, $1_i = 7.6 + 8.1 = 15.7\,kg\,m^2$

Angular speed, $\omega_1 = 300 rev/min$

Angular momentum, $L_i = I_i \omega_i = 15.7 \times 30$ ………………(i)

Moment of inertia for folded hands to a distance of $20\,cm$

$2 \times mr^2$

$= 2 \times 5(0.2)^2 = 0.4\ kg\ m^2$

Final moment of inertia, $I_F = 7.6 + 0.4 = 8\,kg\,m^2$

Final angular speed $= \omega$

Final angular momentum, $L = I_f \omega_f = 0.79\omega_F$ ……………………(ii)

From the conservation of angular momentum, we know: $I_i \omega_i = I_F \omega_F$

$$\therefore \omega_f = \frac{15.7 \times 30}{8} = 58.88 rev/min$$

(b) Kinetic energy is not stored in the given process. In fact, when decrease the moment of inertia, kinetic energy will be increases. The additional kinetic energy comes from the work done by the man to fold his hands attract him

Question 7.24: A bullet of mass $10\,\mathrm{g}$ and speed $500\,\mathrm{m/s}$ is fired into a door and gets embedded exactly at the centre of the door. The door is $1.0\,\mathrm{m}$ wide and weighs $12\,\mathrm{kg}$. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.

(Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$ Answer:
Mass of the bullet, $m = 10\,\mathrm{g} = 10 \times 10^{-3}\,\mathrm{kg}$

Velocity of the bullet, $v = 500\,\mathrm{m/s}$

Thickness of the door, $L = 1\mathrm{m}$

Radius of the door, $r = \frac{1}{2}\mathrm{m}$

Mass of the door, $M = 12\,\mathrm{kg}$

Angular momentum for the bullet on the door:

$\alpha = mvr$

$= \left(10 \times 10^{-3}\right) \times (500) \times \frac{1}{2} = 2.5\,\mathrm{kg\,m^2\,s^{-1}}$ ……………………..(i)

Moment of inertia for the door:

$I = \frac{1}{3}ML^2$

$= \frac{1}{3} \times 12 \times (1)^2 = 4\,\mathrm{kg\,m^2}$

But $\alpha = I\omega$

$\therefore \omega = \frac{\alpha}{I}$

$= \frac{2.5}{4} = 0.625\ \mathrm{rad\ s^{-1}}$

Question 7.25: Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less

than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$

Answer: (a) Moment of inertia of disc $I = I_1$

Angular speed of disc $\mathrm{I} = \alpha$

Angular speed of disc II $= I_2$

Angular momentum of disc II $= \omega_1$

Angular momentum of disc $\mathrm{I}, L_9 = I_1\sigma_1$

Angular momentum of disc II, $L_2 = I_2\omega_2$

Total initial angular momentum, $L_1 = I_1\omega_1 + I_2\omega_2$

When the two discs are joined together, then there is necessary that moments of inertia get added up.

Moment of inertia of two discs, $I = I_1 + I_2$

Let ω be the angular speed of the system.

Total final angular momentum, $L_\gamma = (I_1 + I_2)\omega$

On Using the law of conservation of angular momentum, we have: $L_i = L_\gamma$

$$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$
$$\therefore \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(b) Kinetic energy of disc I, $E_1 = \frac{1}{2}I_1\omega_1^2$

Kinetic energy of disc II, $E_2 = \frac{1}{2}I_2\omega_2^2$

Total initial kinetic energy, $E_i = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2\right)$

When the discs are joined, their moments of inertia will be get added up.

Moment of inertia of the system, $I = I_1 + I_2$

Angular speed of the system $= \omega$

Final kinetic energy $\mathrm{E_f}$

$$= \frac{1}{2}(I_1 + I_2)\omega^2$$

$$= \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2 = \frac{1}{2}\frac{(I_1\omega_1 + I_2\omega_2)^2}{I_1 + I_2}$$

$$E_1 - E_1 = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - \frac{(I_1\omega_1 + I_2\omega_2)^2}{2(I_1 + I_2)}$$

$$= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}\left(\frac{I_1^2\omega_1^2}{I_1 + I_2}\right) - \frac{1}{2}\frac{I_2^2\omega_2^2}{(I_1 + I_2)} - \frac{1}{2}\frac{2I_1I_2\omega_1\omega_2}{(I_1 + I_2)}$$

$$= \frac{1}{(I_1 + I_2)}\left[\frac{1}{2}I_1^2\omega_1^2 + \frac{1}{2}I_1I_2\omega_2^2 + \frac{1}{2}I_1I_2\omega_2^2 + \frac{1}{2}I_2^2\omega_2^2 - \frac{1}{2}I_1^2\omega_1^2 - \frac{1}{2}I_2^2\omega_2^2 - I_1I_2\omega_1\omega_2\right]$$

$$= \frac{I_1I_2}{2(I_1 + I_2)}\left[\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2\right]$$

$$= \frac{I_1I_2(\omega_1 - \omega_2)^2}{2(I_1 + I_2)}$$

All the quantities on right hand side are positive.

$\therefore E_i - E_f > 0$

$E_i > E_f$

The loss of K.E. can be taken as the frictional force that comes into play when the two discs contact with each other.

Question 7.26: Prove the theorem of perpendicular axes.

(Hint: Square of the distance of a point (x, y) in the $x - y$ plane from an axis through the origin perpendicular to the plane is $(x^2 + y^2)$

Prove the theorem of parallel axes

(Hint: If the centre of mass is chosen to be the origin $\sum m_i r_i = 0$

Answer: (a) The theorem of perpendicular axes says that that the moment of inertia of a planar body (lamina) along an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and they also be lie in the plane of the body.

A body with centre O and a point mass m, in the $x - y$ plane at (x, y).

Moment of inertia about $x - axis,\ I_x = mx^2$

Moment of inertia about $y - axis,\ I_y = my^2$

Moment of inertia about z-axis, $I_z = m\left(\sqrt{x^2+y^2}\right)^2$

$$I_x + I_y = mx^2 + my^2$$
$$= m\left(x^2+y^2\right)$$
$$= m\left(\sqrt{x^2+y^2}\right)^2$$
$$I_x + I_y = I_x$$

Hence, the theorem is proved.

(b) The theorem of parallel axes define as that the moment of inertia of a body along any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

Suppose a solid body is made up of n particles, having masses $m_1, m_2, m_3, \ldots, m_{n,}$, at perpendicular distances $r_1, r_2, r_3, \ldots, r_n$ lie from the centre of mass O of the rigid body.

The moment of inertia along axis RS passing through the point O

$$I_{RS} = \sum_{i=1}^{n} mr_i^2$$

The perpendicular distance of mass m_1 from the axis $QP = a + r$

Hence, the moment of inertia about axis QP:

$$I_{QP} = \sum_{i=1}^{n} m_i \left(a + r_i\right)^2$$
$$= \sum_{i=1}^{n} m_i \left(a^2 + r_i^2 + 2ar_i\right)^2$$
$$= \sum_{i=1}^{n} ma^2 + \sum_{i=1}^{n} m_i r_i^2 + \sum_{i=1}^{n} m_i 2ar_i$$
$$= I_{RS} + \sum_{i=1}^{n} m_i a^2 + 2\sum_{i=1}^{2} m_i ar_i^2$$

Now, at the centre of mass, the moment of inertia of all the particles along the axis passing through the centre of mass, which is zero, that is,

$$2\sum_{i=1}^{n} m_i ar_i = 0$$
$$\because a \neq 0$$
$$\sum m_i r_i = 0$$

Also,

$\sum_{i=1}^{n} m_i = M$; M = Total mass of the rigid body

$\therefore I_{QP} = I_{RS} + Ma^2$

Question 7.27: Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height is h given by

$$v^2 = \frac{2gh}{\left(1 + k^2/R^2\right)}$$

Using dynamical consideration (i.e., by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Answer: A body rolling on an inclined plane of height h.

$m =$ Mass of the body

$R =$ Radius of the body

$K =$ Radius of gyration of the body

$v =$ Translational velocity of the

Body $h =$ Height of the inclined

Plane $g =$ Acceleration due to gravity

Total energy from the top of the plane, $E_1 = mgh$

Total energy from the bottom of the plane,

$$E_b = KE_{rot} + KE_{trans} = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

But $I = mk^2$ *and* $\omega = \frac{v}{R}$

$$E_b = \frac{1}{2}\left(mk^2\right)\left(\frac{v^2}{R^2}\right) + \frac{1}{2} mv^2$$

$$= \frac{1}{2} mv^2 \frac{k^2}{R^2} + \frac{1}{2} mv^2$$

$$= \frac{1}{2} mv^2 \left(1 + \frac{k^2}{R^2}\right)$$

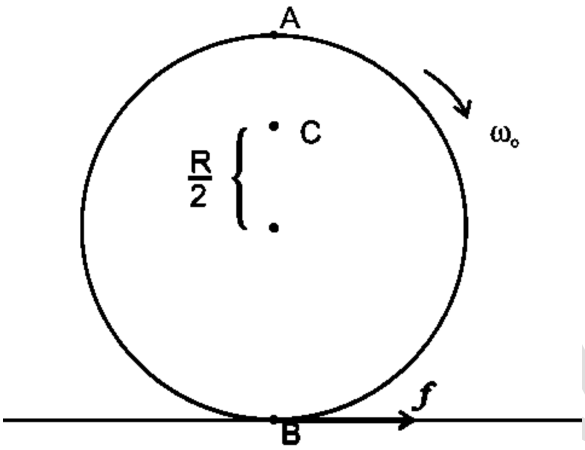
By the law of conservation of energy, we have:

$E_1 = E_b$

$$mgh = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

$$\therefore v = \frac{2\,gh}{\left(1 + k^2/R^2\right)}$$

Question 7.28: A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig. ? Will the disc roll in the direction indicated?



Answer: $v_A = R\omega_0; v_B = R\omega_0; \quad v_e = \left(\frac{R}{2}\right)\omega_0$ The disc will not roll

Angular speed of the disc ω_0

Radius of the disc $= R$

On Using this for linear velocity, $v = \omega_0 R$

For point A:

$v_A = R\omega_0$ in the direction tangential to the right

For point B:

$v_B = R\omega_0$ in the direction tangential to the left

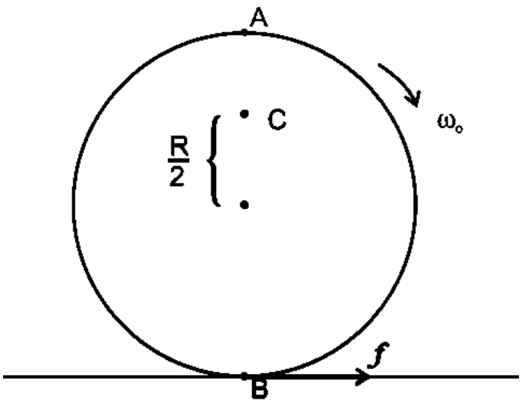
For point C:

$$v_c = \left(\frac{R}{2}\right)\omega_\circ$$

The directions of motion's points A, B, and C on the disc.

Since the disc is placed on that table without friction table, it will not move. This is because the presence of friction is necessary for the rolling of a body.

Question 7.29: Explain why friction is necessary to make the disc in as according to given Fig. roll in the direction indicated.



Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

What is the force of friction after perfect rolling begins?

Answer: A torque is required to move the given disc. As by the definition of torque, the rotating force will be required tangential to the disc. Since the frictional force at point B is about the tangential force at point A . A frictional force is essential for making the disc roll.

Force of friction lie opposite to the direction of velocity on point B. linear velocity direction at point B is tangentially leftward. Hence, frictional force will be lie tangentially rightward. Frictional torque before the start of perfect rolling is perpendicular of the plane of the disc in the outward direction.

Since frictional force lie opposite to the direction of velocity at point B, perfect rolling will start when the velocity will be at that point becomes equal to zero. It will make the frictional force acting on the disc zero.

Question 7.30: A solid disc and a ring, both of radius $10\,\text{cm}$ are placed on a horizontal table simultaneously, with initial angular speed equal to $10\pi\,\text{rad}\,\text{s}^{-1}$. Which of the two will start to roll earlier? The co-efficient of kinetic friction is

$\mu_k = 0.2$

Answer: Radii of the ring and the disc, $r = 10\,\text{cm} = 0.1\,\text{m}$

Initial angular speed, $\omega_0 = 10\pi\,\text{rad}\,\text{s}^{-1}$

Coefficient of kinetic friction, $\mu_k = 0.2$

Initial velocity of both the objects, $u=0$

Motion of the two objects is due to frictional force. According to Newton's second law of motion, the frictional force is, $f=ma\mu_k mg=ma$ Where, $a=$Acceleration induced in the objects $m=$Mass

$a=\mu g$ ……………(i)

According to the first equation of motion, the final velocity of the objects can be as:

$v=u+at$

$=0+\mu gt$

$=\mu gt$ ………………………(ii)

The torque exerted by the frictional force will act in perpendicularly outward direction and the reason of reduction in the initial angular speed.

Torque, $\tau=-l\alpha$

$\alpha=$ Angular acceleration

$\mu_x mgr=-l\alpha$

$\therefore \alpha=\frac{-\mu_2 mgr}{l}$ ………………………….(iii)

By Using the first equation of rotational motion for obtaining the final angular speed:

$e=\omega_0+\alpha t$

$=\omega_0+\frac{-\mu_1 mgr}{l}I$ …………………….(iv)

Rolling starts when linear velocity, $v=r\omega$

$\therefore v=r\left(\omega_0-\frac{\mu_1 gmrt}{I}\right)$………………………….(v)

Equating equations (ii) and (v), we get:

$\mu_k gt=r\left(\omega_b-\frac{\mu_1 gmrt}{I}\right)$

$=r\omega_0-\frac{\mu_1 gmr^2 t}{I}$…………………..(vi)

For the ring: $I=mr^2$

$$\therefore \mu_k g = r\omega_0 - \frac{\mu_k gmr^2 t}{mr^2}$$

$$= r\omega_0 - \mu_k \ gmt_k$$

$$2\mu_k gt = r\omega_0$$

$$\therefore t_r = \frac{r\omega_0}{2\mu_k g}$$

$$= \frac{0.1 \times 10 \times 3.14}{2 \times 0.2 \times 9.8} = 0.80\text{s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)}$$

For the disc: $I = \frac{1}{2}mr^2$

$$\therefore \mu_k gt_4 = r\omega_0 \frac{\mu_k gmr^2 t}{\frac{1}{2}mr^2}$$

$$= r\omega_0 - 2\mu_2 g$$

$$\therefore t_d = \frac{r\omega_2}{3\mu_1 g}$$

$$= \frac{0.1 \times 10 \times 3.14}{3 \times 0.2 \times 0.8} = 0.53\text{s} \ldots\ldots\ldots\ldots\text{(viii)}$$

Since $t_d > t_r$, the disc will start rotating before the ring.

Question 7.31: A cylinder of mass $10\,\text{kg}$ and radius $15\,\text{cm}$ is rolling perfectly on a plane of inclination 30°. The coefficient of static friction $\mu_k = 0.25$ How much is the force of friction acting on the cylinder?

What is the work done against friction during rolling?

If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?

Answer: Mass of the cylinder, $m = 10\,\text{kg}$

Radius of the cylinder, $r = 15\,\text{cm} = 0.15\,\text{m}$

Co-efficient of kinetic friction, $\mu_k = 0.25$

Angle of inclination, $\theta = 30^\circ$

Moment of inertia of a solid cylinder along its geometric axis, $I = \frac{1}{2}mr^2$.

The acceleration of the cylinder

$$a = \frac{mg \sin \theta}{m + \frac{l}{r^2}}$$

$$= \frac{mg \sin \theta}{m + \frac{1mr^2}{2r^2}} = \frac{2}{3} g \sin 30^\circ$$

$$= \frac{2}{3} \times 9.8 \times 0.5 = 3.27 \text{ m/s}^2$$

On Using Newton's second law of motion, we can write net force as:

$$f_{net} = ma$$

$$\sin 30^\circ - f = ma$$

$$f = mg \sin 30^\circ - ma$$

$$= 10 \times 9.8 \times 0.5 - 10 \times 3.27$$

$$= 49 - 32.7 = 16.3 \text{ N}$$

During rotating, the nearest point of contact with the plane comes to rest. So, the work done against to the frictional force is zero.

For rolling without stud, we have the relation:

$$\mu = \frac{1}{3} \tan \theta$$

$$\tan \theta = 3\mu = 3 \times 0.25$$

$$\therefore \theta = \tan^{-1}(0.75) = 36.87^\circ$$

Question 7.32: Read each statement below carefully, and state, with reasons, if it is true or false.

During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.

The instantaneous speed of the point of contact during rolling is zero.

The instantaneous acceleration of the point of contact during rolling is zero.

For perfect rolling motion, work done against friction is zero.

A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Answer: False

Frictional force exist oppose to the direction of motion of the centre of mass of a body. In the case of rolling, the centre of mass is backward in the case of the direction of motion. Hence, frictional force lies in the forward direction.

True

Rolling can be assumed as the rotation of a body about an axis enormous through the point of contact of the body with the ground. Hence, its quickly speed is zero.

False

When a body is rolling, its spontaneous acceleration is not equal to zero.

True

When perfect rolling start, the frictional force lies at the lowermost point which becomes zero. Hence, the work done oppose to friction is also zero.

True

The rolling of a body lies when a frictional force lies between the body and the surface. This frictional force gives the torque essential for rolling. In the absence of a frictional force, the body slip down from the inclined plane under the effect of its weight.

Question 7.33: Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:

Show $\mathrm{p}_i = \mathrm{p}'_i + m_i \mathrm{V}$ **Where** p_i **is the momentum of the** i^{th} **particle (of** m_i **) and** $\mathrm{p}_i = \mathrm{p}'_i + m_i v'_i$**, is the velocity of the** i^{th} **particle relative to the centre of mass.**

Also, prove using the definition of the centre of mass $\sum p'_i = 0$

Show $K = K' + 1/2MV^2$

Where K is the total kinetic energy of the system of particles, K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $1/2MV^2$ **is the kinetic energy of the translation of the system as a whole (i.c. of the centre of mass motion of the system). The result has been used in Sec. 7.14.**

Show $L = \mathbf{L}' + \mathbf{R} \times \mathbf{MV}$

$$\mathbf{L}' = \sum r'_i \times \mathbf{p}'_i$$

Where with velocities taken relative to the centre of mass. Remember $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$ **; rest of the notation is the standard notation used in the chapter. Note** $\mathbf{L}'$ **and** $M\mathbf{R} \times \mathbf{V}$ **can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.**

Show $\frac{d\mathbf{L}'}{dt} = '_i \times \frac{d}{dt}(\mathbf{p}'_i)$

Further show that

$$\frac{d\mathbf{L}'}{dt} = \tau'_{ext}$$

Where τ'_{ext} **is the sum of all external torques acting on the system about the centre of mass?**

(**Hint: Use the definition of centre of mass and Newton's Third Law. Assume the internal forces between any two particles act along the line joining the particles.)**

Answer: (a) Take a system of i moving particles.

Mass of the i^{th} particle $= m_i$

Velocity of the i^{th} particle $= \mathbf{v}_i$

Momentum of the i^{th} particle, $\mathbf{p}_i = m_i \mathbf{v}_i$

Velocity of the centre of mass $= \mathbf{V}$

The velocity of the i^{th} particle w.t.h centre of mass of the system

as: $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}$ ....(i)

Multiplying by equation (i), by m_i equation (i), we get:

$m_i \mathbf{v}'_i = m_i \mathbf{v}_i - m_i$

$\mathbf{p}' i = \mathbf{p} i - m_i \mathbf{V}$

Where,

$\mathrm{p}'_\mathrm{i} = m_i \mathbf{v}_i' =$ Momentum of the i^{th} particle according to the centre of mass of the

system: $\mathbf{p}_i = m_i \mathbf{v}'_i$

We have the relation: $\mathrm{p}'_\mathrm{i} = m_i \mathbf{v}_i' =$

Taking the sum of momentum of all the particles with respect to the centre of mass of the system, we get:

$$\sum_\mathrm{i} \mathrm{p}'_\mathrm{i} = \sum_\mathrm{i} \mathrm{m}_\mathrm{i} \mathrm{v}'_\mathrm{i} = \sum_\mathrm{i} m_\mathrm{i} \frac{d\mathbf{r}_i'}{dr}$$

Where,

$\mathrm{r}'_\mathrm{i} =$ Position vector of i^{th} particle with respect to the centre of mass

$$\mathrm{v}'_\mathrm{i} = \frac{d\mathbf{r}_i'}{dr}$$

According to the definition of the centre of mass, we have:

$$\sum_\mathrm{i} \mathrm{m}_\mathrm{i} \mathrm{r}'_\mathrm{i} = 0$$

$$\sum_\mathrm{i} \mathrm{m}_\mathrm{i} \mathrm{r}'_\mathrm{i} \frac{d\mathbf{r}_i'}{dr} = 0$$

$$\sum_i p'_i = 0\frac{d\mathbf{r}_i'}{dr}$$

We have the relation velocity of the i^{th} particle as:

$$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{v}$$

$$\sum_i m_i v_i = \sum_i m_i v'_i + \sum_i m_i V \text{........................(ii)}$$

Doing the dot product of equation (ii) with itself, we get:

$$\sum_i m_i v_i \sum_i m_i = \sum_i m_i (v_i + v) \sum_i m_i (v_i + v)$$

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i^2 + M^2 \sum_i v_i v' + M^2 \sum_i v_i v_1 + M^2 v^2$$

Here, then get the centre of mass of the system of particles, $\sum_i v_i v'_i = -\sum_i v'_i . v_i$

$$K = K' + \frac{1}{2}MV^2$$

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i^2 + M^2 V^2$$

$$\frac{1}{2}M^2 \sum_i v_i^2 = \frac{1}{2}M^2 \sum_i v_i^2 + \frac{1}{2}M^2 V^2$$

Where,

$K = \frac{1}{2}M\sum v_i^2$ Total kinetic energy for the system of particles

$K' = \frac{1}{2}M\sum v_i^2$ = Total kinetic energy for the system of particles with respect to the centre of mass

$\frac{1}{2}MV^2 =$ Kinetic energy for the translation of the system as a whole

Position vector for the i^{th} particle with respect to origin is r_i

Position vector for the i^{th} particle with respect to the centre of Position vector of the centre of mass along to the origin = R

It is given that $r'_i = r_i$

$Rr_i = r'_i + R$ We have from part (a), p_i

$= p_i + m_i V$

doing the cross product of this relation by r'_i we get:

$$\sum_i r'_i \times p'_i = \sum_i r'_i \times p'_i + \sum_i r'_i \times m_i V$$

$$L = \sum_i \left(r'_i + R\right) \times p'_i + \sum_i \left(r'_i + R\right) \times m_i V$$

$$= \sum_i r'_i \times p'_i + \sum R \times p'_i + \sum_i r'_i \times m_i V + \sum R \times m_i V$$

$$= L' + \sum R \times p'_i + \sum_i r'_i \times m_i V + \sum_i r'_i \times m_i V$$

Where,

$$R \times \sum p'_i = 0 \text{ and } \left(\sum_i r'_i\right) \times M = 0$$

$$\sum_i m_i M = M$$

$$\therefore L = L' + R \times MV$$

We get the relation:

$$L' = \sum_i r'_i \times p'_i$$

$$\frac{dL'}{dt} = \frac{d}{dt}\left(\sum_i r'_i \times p'_i\right)$$

$$= \frac{d}{dt}\left(\sum_i r'_i\right) \times p'_i + \sum_i r'_i \times \frac{d}{dt}\left(p'_i\right)$$

$$= \frac{d}{dt}\left(\sum_i m_i r'_i\right) \times v'_i + \sum_i r'_i \times \frac{d}{dt}\left(p'_i\right)$$

Where, r'_i, is the position vector along to the centre of mass of the system of particles.

$$\therefore \sum_i m_i r'_i = 0$$

$$\frac{dL'}{dt} = \sum_i r'_i \times \frac{d}{dt}\left(p'_i\right)$$

We have the relation:

$$\frac{d\mathbf{L}'}{dt} = \sum_i r'_i \times \frac{d}{dt}\left(\mathbf{p}'\right)$$

$$= \sum_i r'_i \times m_i \frac{d}{dt}\left(\mathbf{v}'_i\right)$$

Where, $\frac{d}{dt}\left(v'_i\right)$ is the rate of change of velocity of the i^{th} particle along to the centre of mass of the system

Therefore, now using Newton's third law of motion, we can write:

$m_1 \frac{d}{dt}\left(v_i'\right) = \text{External force lie on the particle } = \sum_i^n \tau'_i$

$i.e. \sum \mathbf{r}', \times m, \frac{d}{dt}\left(\mathbf{v}';\right) = \tau'_{ext} = \text{External torque lies on the system as a whole}$

$\therefore \frac{d\mathbf{L}'}{dt} = r'_{ext}$