Chapter-8: Gravitation

EXAMPLES

8.1: Let the speed of the planet at the perihelion P in Fig. $8.1(a)$ be v_P and the Sun-planet distance SP be $r_{P,}$. Relate $\{r_{P,} v_P\}$ to the corresponding quantities at the aphelion $\{r_{A,} v_A\}$. Will the planet take equal times to traverse BAC and CPB?

Solution: Angular Momentum at P: $L_p = m_p r_p v_p$

Since $r_{P,}$ & v_P are mutually perpendicular

Similarly: $L_A = m_p r_A v_A$

By angular momentum conservation: $m_p r_p v_p = m_p r_A v_A$

$$\Rightarrow \frac{v_p}{v_A} = \frac{r_A}{r_p}$$

As $r_A > r_p$, $v_p > v_A$.

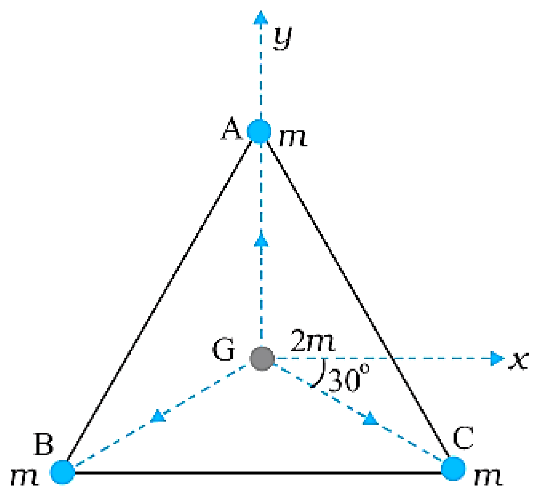
SBAC is larger than SBPC because it is circumscribed by an ellipse and the radius vectors SB and SC. So, from Kepler's second law, equal areas are swept in equal times.

Equal regions are swept in equal periods, according to Kepler's second law.

8.2 Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.

(a) What is the force acting on a mass $2m$ placed at the centroid G of the triangle?

Solution: (a)The angle formed by GC and the positive x-axis, as well as the angle formed by GB and the negative x-axis, is $30°$.



Individual forces in vector notation are: $\mathbf{F}_{GA} = \frac{Gm(2m)}{1}\hat{\mathbf{j}}$

$$\Rightarrow \quad \mathbf{F}_{GB} = \frac{Gm(2m)}{1}\left(-\hat{\mathbf{i}}\cos 30^\circ - \hat{\mathbf{j}}\sin 30^\circ\right)$$

$$\Rightarrow \quad \mathbf{F}_{GC} = \frac{Gm(2m)}{1}\left(+\hat{\mathbf{i}}\cos 30^\circ - \hat{\mathbf{j}}\sin 30^\circ\right)$$

Using the principle of superposition and the law of vector addition. The resultant gravitational force $\mathbf{F}_R$ on $(2m)$ is: $\mathbf{F}_R = \mathbf{F}_{CA} + \mathbf{F}_{GB} + \mathbf{F}_{GC}$

$$\mathbf{F}_R = 2Gm^2\ \mathbf{j} + 2Gm^2\left(-\hat{\mathbf{i}}\cos 30^\circ - \mathbf{j}\sin 30^\circ\right) + 2Gm^2\left(\hat{\mathbf{i}}\cos 30^\circ - \hat{\mathbf{j}}\sin 30^\circ\right) = 0$$

On the other hand, one may expect the resultant force to be zero due to symmetry.

(b) What is the force if the mass at the vertex A is doubled? (Take AG = BG = CG = 1 m)

(b) If the mass at vertex A is doubled, the result is : $F'_{GA} = \frac{G2m \cdot 2m}{1}\hat{j} = 4Gm^2\hat{j}$

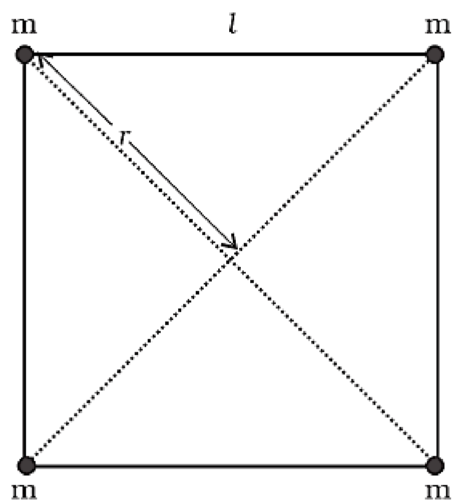
$$\Rightarrow \quad F'_{GB} = F_{GB} \text{ and } F'_{GC} = F_{GC}$$

$$\Rightarrow \quad F'_R = F'_{GA} + F'_{GB} + F'_{GC}$$

$$\Rightarrow \quad F'_R = 2Gm^2\hat{j}$$

8.3: Find the potential energy of a system of four particles placed at the vertices of a square of side l Also obtain the potential at the centre of the square.

Solution: Consider four masses, each with a mass of m, at the four corners of a square with a side length of l.



Ata distance of l, we have four mass pairs and two diagonal pairs at a distance of $2l$

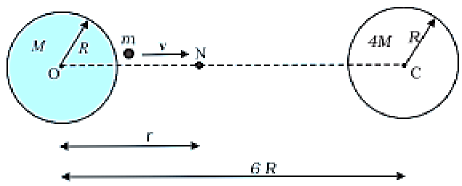
Hence, $W(r) = -4\frac{Gm^2}{l} - 2\frac{Gm^2}{\sqrt{2}l}$.

$$W(r) = -\frac{2Gm^2}{l}\left(2 + \frac{1}{\sqrt{2}}\right) = -5.41\frac{Gm^2}{l}$$

Now, gravitational potential at the centre of the square $(r = \sqrt{2}l/2)$ is:

$$U(r) = -4\sqrt{2}\frac{\text{Gm}}{l}$$

8.4 Two uniform solid spheres of equal radii R, but mass M and 4 M have a centre-to-centre separation 6 R, as Shown. The two spheres are held fixed. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.



Solution: The projectile is acted on by two spheres mutually opposed gravitational forces. The neutral point N is the point at which the two forces perfectly cancel each other out.

If ON = r:

$$\frac{GMm}{r^2} = \frac{4GMm}{(6R-r)^2}$$

$\Rightarrow \quad (6R-r)^2 = 4r^2$

$\Rightarrow \quad r = 2R \quad \text{or} -6R$

In this case, the neutral point $r = -6R$ is irrelevant. As a result, ON = r = $2R.$ It is enough to project the particle at a speed that will allow it to reach N. the greater gravitational pull of $4M$ would suffice.

Mechanical energy: $E_i = \frac{1}{2}mv^2 - \frac{GMm}{R} - \frac{4GMm}{5R}$

The speed approaches zero at the neutral point N.

Mechanical energy at N: $E_N = -\frac{GMm}{2R} - \frac{4GMm}{4R}$

By the principle of conservation of mechanical energy: $\frac{1}{2}v^2 - \frac{GM}{R} - \frac{4GM}{5R} = -\frac{GM}{2R} - \frac{GM}{R}$

$$v^2 = \frac{2GM}{R}\left(\frac{4}{5} - \frac{1}{2}\right)$$

$$v = \left(\frac{3GM}{5R}\right)^{1/2}$$

It's worth noting that the projectile's speed is zero at N, but nonzero when it hits the heavier sphere $4M$.

8.5: The planet Mars has two moons, phobos and delmos.

(i) phobos has a period 7 hours, 39 minutes, and an orbital radius of 9.4×10^3 km. Calculate the mass of mars.

Solution: (i) Using the equation: $T^2 = \frac{4\pi^2}{GM_m}R^3$ (with the sun's mass replaced by the Martian mass M_m)

$$\mathrm{M}_m = \frac{4\pi^2}{G}\frac{R^3}{T^2} = \frac{4\times(3.14)^2\times(9.4)^3\times10^{18}}{6.67\times10^{-11}\times(459\times60)^2}$$

$$M_m = \frac{4\times(3.14)^2\times(9.4)^3\times10^{18}}{6.67\times(4.59\times6)^2\times10^{-5}} = 6.48\times10^{23}\,\text{kg.}$$

(ii) Assume that earth and mars move in circular orbits around the sun, with the Martian orbit being 1.52 times the orbital radius of the earth. What is the length of the Martian year in days?

(ii) From Kepler's third law: $\frac{T_M^2}{T_E^2} = \frac{R_{MS}^3}{R_{ES}^3}$

where R_{MS} is the mars -sun distance and R_{ES} is the earth-sun distance.

$$\therefore T_M = (1.52)^{3/2}\times365$$

$$T_M = 684 \text{ days}$$

Example 8.6 Weighing the Earth: You are given the following data: $\mathrm{g} = 9.81\mathrm{ms}^{-2,}$ $R_E = 6.37\times10^6\,\mathrm{m}$, the distance to the moon $R = 3.84\times10^8\,\mathrm{m}$ and the time period of the moon's revolution is 27.3 days. Obtain the mass of the Earth $\mathrm{M_E}$ in two different ways.

Solution: Using Equation: $M_E = \frac{gR_E^2}{G}$

$$\Rightarrow \quad M_E = \frac{9.81\times\left(6.37\times10^6\right)^2}{6.67\times10^{-11}}$$

$$\Rightarrow \quad M_E = 5.97\times10^{24}\,\text{kg}$$

By the derivation of Kepler's third law: $T^2 = \frac{4\pi^2 R^3}{GM_E}$

$$M_E = \frac{4\pi^2 R^3}{GT^2} = \frac{4\times3.14\times3.14\times(3.84)^3\times10^{24}}{6.67\times10^{-11}\times(27.3\times24\times60\times60)^2}$$

$$M_E = 6.02\times10^{24}\,\text{kg}$$

Both procedures produce nearly identical results, with the difference between them being less than 1%. **Example 8.7 Express the constant k of eq.** $T^2 = k\left(R_E + h\right)^3$ $\left(\text{ where } k = 4\pi^2/GM_E\right)$ **in days and kilometres. Given** $k = 10^{-13}\text{s}^2\text{m}^{-3}$. **The moon is at a distance of 3.84 × 10^5 km from the earth. Obtain its time-period of revolution in days.**

Solution: Given: $k = 10^{-13}\text{s}^2\text{m}^{-3}$

$$k = 10^{-13}\left[\frac{1}{(24\times60\times60)^2}\text{d}^2\right]\left[\frac{1}{(1/1000)^3\text{km}^3}\right] = 1.33\times10^{-14}\text{d}^2\text{km}^{-3}$$

Using Eq. $T^2 = k\left(R_E + h\right)^3$ $\left(\text{ where } k = 4\pi^2/GM_E\right)$ and the given value of k, the time period of the moon is: $T^2 = \left(1.33\times10^{-14}\right)\left(3.84\times10^5\right)^3$

$$T = 27.3\text{d}$$

Example 8.8: A 400 kg satellite is in a circular orbit of radius $2R_E$ about the Earth. How much energy is required to transfer it to a circular orbit of radius $4R_E$? What are the changes in the kinetic and potential energies?

Solution: Initially: $E_i = -\frac{GM_E m}{4R_E}$

Finally: $E_f = -\frac{GM_E m}{8R_E}$

Now, change in the total energy: $\Delta E = E_f - E_i$

$$= \frac{GM_E m}{8R_E} = \left(\frac{GM_E}{R_E^2}\right)\frac{mR_E}{8}$$

$$\Delta E = \frac{gmR_E}{8} = \frac{9.81 \times 400 \times 6.37 \times 10^6}{8} = 3.13 \times 10^9 \text{ J}$$

kinetic energy is reduced, and it mimics: $\Delta E : \Delta K = K_f - K_i = -3.13 \times 10^9 \text{ J}$

The change in potential energy is equal to the total energy change: $\Delta V = V_f - V_i = -6.25 \times 10^9 \text{ J}$

Exercise:

8.1: Answer the following:

(a) You can shield a charge from electrical forces by putting it inside a hollow conductor. Can you shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means?

Solution: (a) No

The gravitational pull of adjacent objects on matter cannot be shielded in any way. This is because, unlike electrical forces, gravitational force is unaffected by the nature of the medium. It is also unaffected by the status of other items.

(b) An astronaut inside a small spaceship orbiting around the earth cannot detect gravity. If the space station orbiting around the earth has a large size, can he hope to detect gravity?

(b) Yes

The astronaut will be able to detect the change in Earth's gravity if the space station is large enough (g).

(c) If you compare the gravitational force on the earth due to the sun to that due to the moon, you will find that the Sun's pull is greater than the moon's pull. (You can check this yourself using the data available in the succeeding exercises). However, the tidal effect of the moon's pull is greater than the tidal effect of sun. Why?

(c)

The tidal effect is proportional to the distance cubed, whereas gravitational force is proportional to the distance squared. The tidal effect of the Moon's pull is stronger than the tidal effect of the Sun's pull because the distance between the Moon and the Earth is smaller than the distance between the Sun and the Earth.

8.2: Choose the correct alternative:

(a) Acceleration due to gravity increases/decreases with increasing altitude.

(b) Acceleration due to gravity increases/decreases with increasing depth (assume the earth to be a sphere of uniform density).

I Acceleration due to gravity is independent of mass of the earth/mass of the body.

(d) The formula $-GMm\left(1/r_2-1/r_1\right)$ **is more/less accurate than the formula** $mg\left(r_2-r_I\right)$ **for the difference of potential energy between two points** $\mathbf{r_2}$ **and** $\mathbf{r_1}$ **distance away from the centre of the earth.**

Solution:

a) Decreases
b) Decreases
c) Mass of the body
d) More

Explanation: The relationship gives the acceleration due to gravity at height: $g_h=\left(1-\frac{2h}{R_e}\right)g$

Here, R_e = Radius of the earth

$\mathbf{2.5\times10^{11}}$ = Acceleration due to gravity

The given relationship shows that as height increases, the acceleration due to gravity decreases.

The relationship: gives the acceleration due to gravity at depth d:

$$g_d=\left(1-\frac{d}{R_e}\right)g$$

The given relationship shows that as depth increases, the acceleration due to gravity decreases.

The relation gives the acceleration due to gravity of a body of mass m : $g=\frac{GM}{R^2}$

where, G = Universal gravitational constant

$M=$ Mass of the Earth

$R=$ Radius of the Earth

As a result, it may be deduced that gravity’s acceleration is independent of the body’s mass.

The gravitational potential energy of two sites separated by r_2 and r_1 from the Earth's centre is given by: $V\left(r_1\right)=-\frac{GmM}{r_1}\ \&\ V\left(r_2\right)=-\frac{GmM}{r_2}$

Difference in potential energy: $V=V\left(r_2\right)-V\left(r_1\right)=-GmM\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$

As a result, this formula is more precise than the formula

$mg\left(r_2 - r_I\right)$

8.3: Suppose there existed a planet that went around the sun twice as fast as the earth. What would be its orbital size as compared to that of the earth?

Solution: Lesser by a factor of 0.63

⇨ Time taken by the Earth to complete one revolution: $T_e = 1$ year

⇨ Orbital radius of the Earth: $R_e = 1\text{AU}$

⇨ Time taken by the planet to complete one revolution: $T_P = \frac{1}{2}T_e = \frac{1}{2}$ year

⇨ Orbital radius of the planet $= R_p$

From Kepler's third law: $\left(\frac{R_r}{R_e}\right)^3 = \left(\frac{T_r}{T_e}\right)^2$

⇨ $\frac{R_p}{R_e} = \left(\frac{T_p}{T_e}\right)^{\frac{2}{3}}$

⇨ $\frac{R_p}{R_e} = \left(\frac{\frac{1}{2}}{1}\right)^{\frac{2}{3}} = (0.5)^{\frac{2}{3}} = 0.63$

As a result, the planet's orbital radius will be 0.63 times less than that of the Earth.

8.4: I_0, one of the satellites of Jupiter, has an orbital period of 1.769 days and the radius of the orbit is 4.22×10^8 m. Show that the mass of Jupiter is about one-thousandth that of the sun.

Solution: Orbital period: $I_0, T_{b_u} = 1.769 \text{ days} = 1.769 \times 24 \times 60 \times 60\text{s}$

Orbital radius: $I_0, R_{le} = 4.22 \times 10^8 \text{m}$

Satellite I_0 is revolving around the Jupiter

Mass of the latter is given by: $M_j = \frac{4\pi^2 R_t^3}{GT_{lo}^2} \ldots(1)$

Where, M_j = Mass of Jupiter

G = Universal gravitational Constant

Orbital period of the Earth: $T_e = 365.25 \text{ days} = 365.25 \times 24 \times 60 \times 60\text{s}$

Orbital radius of the Earth: $R_e = 1\text{AU} = 1.496 \times 10^{11}\,\text{m}$

Mass of the sun: $M_s = \dfrac{4\pi^2 R_c^3}{GT_e^2}$ …(2)

$$\therefore \frac{M_s}{M_j} = \frac{4\pi^2 R_e^3}{GT_e^2} \times \frac{GT_{bo}^2}{4\pi^2 R_{lo}^3} = \frac{R_e^3}{R_b^3} \times \frac{T_b^2}{T_e^2}$$

$$\frac{M_s}{M_j} = \left(\frac{1.769 \times 24 \times 60 \times 60}{365.25 \times 24 \times 60 \times 60}\right)^2 \times \left(\frac{1.496 \times 10^{11}}{4.22 \times 10^8}\right)^3 = 1045.04$$

$$\therefore \frac{M_s}{M_j} \sim 1000$$

$$M_s \sim 1000 \times M_J$$

As a result, Jupiter’s mass is estimated to be around one-thousandth that of the Sun.

8.5: Let us assume that our galaxy consists of 2.5×10^{11} stars each of one solar mass. How long will a star at a distance of $50,000$ ly from the galactic centre take to complete one revolution? Take the diameter of the Milky Way to be 105 ly.

Solution: Mass of our galaxy: $M = 2.5 \times 10^{11}$ solar mass

Solar mass= Mass of sun= $2.0 \times 10^{36}\,\text{kg}$

Mass of our galaxy: $M = 2.5 \times 10^{11} \times 2 \times 10^{36} = 5 \times 10^{41}\,\text{kg}$

Diameter of Milky Way: $d = 10^5\,\text{ly}$

Therefore, Radius of Milky Way: $r = 5 \times 10^4\,\text{ly}$

Now, $1\text{ly} = 9.46 \times 10^{15}\,\text{m}$

⇨ $r = 5 \times 10^4 \times 9.46 \times 10^{15}$

⇨ $r = 4.73 \times 10^{20}\,\text{m}$

Because a star spins around the Milky Way’s galactic centre, its time period is determined by the relationship: $T = \left(\dfrac{4\pi^2 r^3}{GM}\right)^{\frac{1}{2}}$

$$\Rightarrow T = \left(\frac{4 \times (3.14)^2 \times (4.73)^3 \times 10^{60}}{6.67 \times 10^{-11} \times 5 \times 10^{41}}\right)^{\frac{1}{2}} = \left(\frac{39.48 \times 105.82 \times 10^{30}}{33.35}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad = \left(125.27\times10^{30}\right)^{\frac{1}{2}} = 1.12\times10^{16}\,\text{s}$$

Since, $\text{lyear} = 365\times324\times60\times60\text{s}$

Therefore, $1\text{s} = \dfrac{1}{365\times24\times60\times60}$ years

$$\therefore\ 1.12\times10^{16}\,\text{s} = \frac{1.12\times10^{16}}{365\times24\times60\times60}$$

$$= 3.55\times10^{8}\ \text{years}$$

8.6: Choose the correct alternative:

(a) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its kinetic/potential energy.

Solution: (a) Kinetic Energy

A satellite's total mechanical energy is the sum of its kinetic (always positive) and potential energy (may be negative). The gravitational potential energy of the satellite is zero at infinity. Because the Earth-satellite system is a bound system, the satellite's total energy is negative. As a result, at infinity, the total energy of an orbiting satellite equals the negative of its kinetic energy.

(b) The energy required to launch an orbiting satellite out of earth's gravitational influence is more/less than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

(b) Less

A satellite in orbit acquires a specific quantity of energy that allows it to rotate around the Earth. It gets this energy from its orbit. It takes less energy to move out of the gravitational field of the Earth than it does to move out of the gravitational field of a stationary item on the Earth's surface that has no energy at all.

8.7: Does the escape speed of a body from the earth depend on the mass of the body, the location from where it is projected, the direction of projection, the height of the location from where the body is launched?

Solution: No, No, No, Yes

Escape Velocity of the body: $v_{ec} = \sqrt{2gR}$

Where, g = Acceleration due to gravity

R = Radius of earth

The escape velocity is independent of the mass of the body and the direction of its projection, as shown by equation I However, the gravitational potential at the location where the body is launched is a factor. Because this potential is influenced by the height of the point, escape velocity is also influenced by these parameters.

8.8: A comet orbits the Sun in a highly elliptical orbit. Does the comet have a constant (a) linear speed

(b) angular speed,

I angular momentum,

(d) kinetic energy,

I potential energy,

(f) total energy throughout its orbit? Neglect any mass loss of the comet when it comes very close to the Sun.

Solution:

a) No
b) No
c) Yes
d) No
e) No
f) Yes

At all places along a comet's highly elliptical orbit around the Sun, angular momentum and total energy are constant. Its linear, angular, kinetic, and potential energy vary depending on where it is in the orbit.

8.9: Which of the following symptoms is likely to afflict an astronaut in space (a) swollen feet, (b) swollen face, (c) headache, (d) orientational problem?

Solution: (b), (c), and (d)

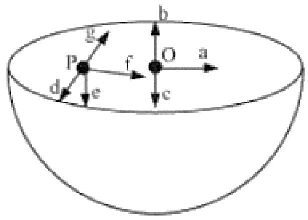
Due to gravitational attraction, the legs hold the full mass of a body in a standing position. Because there is no gravity in space, an astronaut feels weightless. As a result, an astronaut's swollen feet have no bearing on his or her performance in space.

The perceived weightlessness in space causes a bloated face in most people. The face is made up of sense organs such as the eyes, ears, nose, and mouth. An astronaut in orbit may have this condition.

Headaches are brought on by mental tension. It can have an impact on an astronaut's ability to work in space. Different orientations exist in space. As a result, an astronaut in space may experience an orientational difficulty.

8.10: Choose the correct answer from among the given ones:

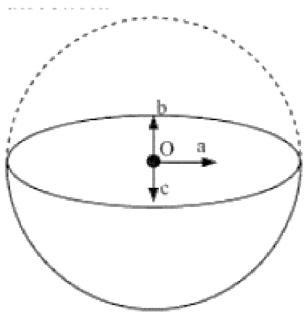
The gravitational intensity at the centre of a hemispherical shell of uniform mass density has the direction indicated by the arrow: (i) a, (ii) b, (iii) c, (iv) O.



Solution: (iii)

In a spherical shell, the gravitational potential (V) is constant at all places. Hence, the gravitational potential gradient $\left(\frac{dV}{dr}\right)$ is zero everywhere inside the spherical shell. The negative of gravitational intensity equals the gravitational potential gradient. As a result, intensity is zero throughout the spherical shell. This shows that gravitational forces at a point in a spherical shell are symmetric.

The net gravitational force exerted on a particle placed at centre O will be downward if the upper half of a spherical shell is cut off (as indicated in the given illustration).



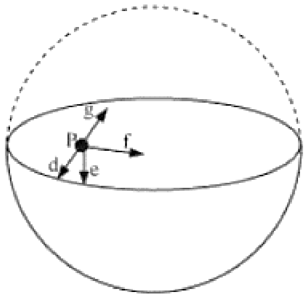
Because gravitational intensity is defined as the gravitational force per unit mass at a given location, it will also act downward. As a result, the gravitational intensity at the centre O of the chosen hemispheric shell follows the arrow c.

8.11: Choose the correct answer from among the given ones:

For the problem 8.10, the direction of the gravitational intensity at an arbitrary point P is indicated by the arrow (i) d, (ii) e, (iii) f, (iv) g.

Solution: Option (ii) is correct.

Gravitational potential (V) is constant at all points in a spherical shell. Hence, the gravitational potential gradient $\left(\frac{dV}{dr}\right)$ is zero everywhere inside the spherical shell. The negative of gravitational intensity equals the gravitational potential gradient. As a result, intensity is zero throughout the spherical shell. This shows that gravitational forces at a point in a spherical shell are symmetric.



Because gravitational intensity is defined as the gravitational force per unit mass at a given location, it will also act downward. As a result, the gravitational intensity at any point P of the hemispheric shell follows the direction indicated by arrow **e**.

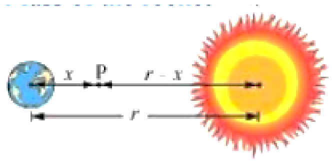
8.12: A rocket is fired from the earth towards the sun. At what distance from the earth's centre is the gravitational force on the rocket zero? Mass of the sun = 2×10^{30} kg , mass of the earth = 6×10^{24} kg. Neglect the effect of other planets etc. (orbital radius = 1.5×10^{11} m).

Solution: Mass of the Sun: $M_s = 2\times10^{30}\,\text{kg}$

Mass of the Earth: $M_e = 6\times10^{24}\,\text{kg}$

Orbital radius: $r = 1.5\times10^{11}\,\text{m}$

Let mass of the rocket = m



Let x be the distance from the Earth's centre at which the gravitational pull on satellite P is zero.

We may equate gravitational forces acting on satellite P under the influence of the Sun and the Earth using Newton's law of gravitation:

$$\Rightarrow \frac{GmM_s}{(r-x)^2} = Gm\frac{M_e}{x^2}$$

$$\Rightarrow \left(\frac{r-x}{x}\right)^2 = \frac{M_3}{M_e}$$

$$\Rightarrow \frac{r-x}{x} = \left(\frac{2\times10^{30}}{60\times10^{24}}\right)^{\frac{1}{2}} = 577.35$$

$$\Rightarrow \begin{array}{l} 1.5\times10^{11} - x = 577.35x \\ 578.35x = 1.5\times10^{11} \end{array}$$

$$\Rightarrow x = \frac{1.5\times10^{11}}{578.35} = 2.59\times10^{8}\,\text{m}$$

8.13: How will you 'weigh the sun', that is estimate its mass? The mean orbital radius of the earth around the sun is $1.5\times10^{8}\,\text{km}$.

Solution: Orbital radius of the Earth: $r=1.5\times10^{11}\,\text{m}$

The length of time it takes the Earth to complete one revolution: $T=1$ year $=365.25$ days

⇨ $T=365.25\times24\times60\times60\text{s}$

Universal gravitational constant, $\text{G}=6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$

Now, mass of the Sun: $M=\dfrac{4\pi^2r^3}{GT^2}=\dfrac{4\times(3.14)^2\times\left(1.5\times10^{11}\right)^3}{6.67\times10^{-11}\times(365.25\times24\times60\times60)^2}$

⇨ $M=\dfrac{133.24\times10}{6.64\times10^4}=2.0\times10^{30}\,\text{kg}$

Hence, the mass of the Sun $2.0\times10^{30}\,\text{kg}$

8.14: A Saturn year is 29.5 times the earth year. How far is the Saturn from the sun if the earth is $1.50\ \times10^{8}$ km away from the sun?

Solution: Distance between the Earth and the Sun: $r_e=1.5\times10^{8}\,\text{km}=1.5\times10^{11}\,\text{m}$.

Time period of Earth: $=T_e$

Time period of Saturn: $T_s=29.5T_e$

Distance of Saturn from the Sun: r_s

Therefore, From Kepler's third law: $T=\left(\dfrac{4\pi^2r^3}{GM}\right)^{\frac{1}{2}}$

For Saturn and Sun: $r_s=r_e\left(\dfrac{T_s}{T_e}\right)^{\frac{2}{3}}$

⇨ $r_s=r_e\left(\dfrac{T_s}{T_e}\right)^{\frac{2}{3}}$

⇨ $r_s=1.5\times10^{11}\left(\dfrac{29.5T_e}{T_e}\right)^{\frac{2}{3}}$

⇨ $r_s=1.5\times10^{11}\times9.55$

⇨ $r_s=14.32\times10^{11}\,\text{m}$

8.15: A body weighs 63 N on the surface of the earth. What is the gravitational force on it due to the earth at a height equal to half the radius of the earth?

Solution: Weight: $W = 63\text{N}$

Acceleration due to gravity at height h: $g' = \dfrac{g}{\left(1+\dfrac{h}{R_e}\right)^2}$

For $h = \dfrac{R_e}{2}$

Where, $Re =$ Radius of the Earth

Now, $g' = \dfrac{g}{\left(1+\dfrac{R_e}{2\times R_e}\right)^2} = \dfrac{g}{\left(1+\dfrac{1}{2}\right)^2} = \dfrac{4}{9}g$

The weight of a body of mass m standing at height h is calculated as

$$W' = mg' = m\times\frac{4}{9}g = \frac{4}{9}\times mg$$

$$W' = \frac{4}{9}\times 63 = 28\text{N}$$

8.16: Assuming the earth to be a sphere of uniform mass density, how much would a body weigh halfway down to the centre of the earth if it weighed 250 N on the surface?

Solution: Weight of a body of mass m: $W = mg = 250\text{N}$

Depth at which body of mass m is located: $d = \dfrac{1}{2}R_e$

Were, $Re =$ Radius of the Earth

Now, Acceleration due to gravity at depth g (d): $g' = \left(1-\dfrac{d}{R_e}\right)g = \left(1-\dfrac{R_e}{2\times R_e}\right)g = \dfrac{1}{2}g$

Weight of body at depth d: $W' = mg' = m\times\dfrac{1}{2}g = \dfrac{1}{2}mg = \dfrac{1}{2}W$

$$W' = \frac{1}{2}\times 250 = 125\text{N}$$

8.17: A rocket is fired vertically with a speed of km s^{-1} from the earth's surface. How far from the earth does the rocket go before returning to the earth? Mass of the earth = 6.0×10^{24} kg ; mean radius of the earth $= 6.4\times10^6\text{m}; \text{G} = 6.67\times10^{-11}\text{Nm}^2\text{kg}^{-2}$ $f_g > f_c$

Solution: Velocity of the rocket: $v = 5\text{km/s} = 5\times10^3\text{m/s}$

Mass of the Earth: $M_e = 6.0\times10^{24}\text{kg}$

Radius of the Earth: $R_e = 6.4\times10^6\text{m}$

Height reached by rocket mass**:** $m = h$

Now, Total energy of the rocket = Kinetic energy + Potential energy

$$= \frac{1}{2}mv^2 + \left(\frac{-GM_e m}{R_e}\right)$$

At highest point h, $v = 0$

Therefore, $\text{Potential energy} = -\frac{GM_e m}{R_e + h}$

$$\text{Total energy of the rocket} = 0 + \left(-\frac{GM_e m}{R_e + h}\right) = -\frac{GM_e m}{R_e + h}$$

From the law of conservation of energy:

At the Earth's surface, the total energy of the rocket = the total energy at a height of h

$$\Rightarrow \frac{1}{2}mv^2 + \left(-\frac{GM_e m}{R_e}\right) = -\frac{GM_e m}{R_e + h}$$

$$\Rightarrow \frac{1}{2}v^2 = GM_e\left(\frac{1}{R_e} - \frac{1}{R_e + h}\right)$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{GM_e h}{R_e(R_e + h)} \times \frac{R_e}{R_e}$$

$$\Rightarrow \frac{1}{2}\times v^2 = \frac{gR_e h}{R_e + h}$$

where, $g = \frac{GM}{R_\ell^2} = 9.8\text{m/s}^2$ (Acceleration due to gravity)

$$\therefore v^2(R_v + h) = 2gRh$$
$$v^2 R_e = h(2gR_y - v^2)$$

$$\Rightarrow h = \frac{R_e v^2}{2gR_e - v^2} = \frac{6.4\times10^6\times(5\times10^3)^2}{2\times9.8\times6.4\times10^6 - (5\times10^3)^2}$$

$$\Rightarrow h = \frac{6.4\times25\times10^{12}}{100.44\times10^6} = 1.6\times10^6\text{m}$$

$$= R_e + h$$

Height achieved by the rocket with respect to the centre of the Earth: $= 6.4\times10^{6} + 1.6\times10^{6}$

$$= 8.0\times10^{6}\,\text{m}$$

8.18: The escape speed of a projectile on the earth's surface is $\mathbf{km\ s^{-1.}}$ A body is projected out with thrice this speed. What is the speed of the body far away from the earth? Ignore the presence of the sun and other planets.

Solution: Escape velocity of a projectile: $v_{\text{esc}} = 11.2\text{km/s}$

Projection velocity: $v_{\text{p}} = 3\text{v}_{\text{esc}}$

Mass of the projectile: $= m$

Velocity of the projectile far away from the Earth: $= v_{\text{f}}$

Now, Total energy on the Earth: $= \frac{1}{2}mv_p^2 - \frac{1}{2}mv_{ec}^2$

The projectile's gravitational potential energy is zero when it is far away from the Earth.

Total energy far away from the Earth: $= \frac{1}{2}mv_f^2$

Now, from the law of conservation of energy: $\frac{1}{2}mv_{\text{p}}^2 - \frac{1}{2}mv_{\text{occ}}^2 = \frac{1}{2}mv_{\text{f}}^2$

$$\Rightarrow \quad v_{\text{f}} = \sqrt{v_{\text{p}}^2 - v_{\text{cs}}^2} = \sqrt{(3v_{\text{esc}})^2 - (v_{\text{cs}})^2}$$

$$v_f = \sqrt{8}v_{\text{csc}}$$

$$\Rightarrow \quad v_f = \sqrt{8}\times 11.2 = 31.68\text{km/s}$$

8.19: A satellite orbits the earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the earth's gravitational influence? Mass of the satellite = 200 kg ; mass of the earth $= 6.0\times10^{24}\,\text{kg}$; radius of the earth $= 6.4\times10^{6}\,\text{m}$;
$\text{G} = 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$

Solution: Mass of the Earth: $M = 6.0\times10^{24}\,\text{kg}$

Mass of the Satellite**:** $m = 200\text{kg}$

Radius of the Earth: $R_{\text{e}} = 6.4\times10^{6}\,\text{m}$

Height of the satellite: $h = 400\text{km} = 4\times10^{5}\,\text{m} = 0.4\times10^{6}\,\text{m}$

$\text{G} = 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$ (Universal gravitational constant)

Total energy: $=\frac{1}{2}mv^2+\left(\frac{-GM_e m}{R_e+h}\right)$

The satellite is tethered to the Earth, as shown by the negative sign. This is referred to as the satellite's bound energy.

Now, Energy required to send the satellite out of its orbit = – (Bound energy)

$$T.E=\frac{1}{2}\frac{GM_e m}{\left(R_e+h\right)}$$

$$T.E=\frac{1}{2}\times\frac{6.67\times10^{-11}\times6.0\times10^{24}\times200}{\left(6.4\times10^6+0.4\times10^6\right)}$$

$$T.E=\frac{1}{2}\times\frac{6.67\times6\times2\times10}{6.8\times10^6}=5.9\times10^9\,\mathrm{J}$$

8.20: Two stars each of one solar mass $\left(=2\times10^{30}\,\mathrm{kg}\right)$ are approaching each other for a head on collision. When they are a distance 109 km , their speeds are negligible. What is the speed with which they collide? The radius of each star is 104 km. Assume the stars to remain undistorted until they collide. (Use the known value of G)

Solution: Mass of each star: $M=2\times10^{30}\,\mathrm{kg}$

Radius of each star: $R=10^4\,\mathrm{km}=10^7\,\mathrm{m}$

Distance between the stars: $r=10^9\,\mathrm{km}=10^{12}\,\mathrm{m}$

$v=0$ total energy of two stars separated at r distance with insignificant speeds

$$=\frac{-GMM}{r}+\frac{1}{2}mv^2$$

$$=\frac{-GMM}{r}+0\ldots(1)$$

Consider the following scenario in which the stars are ready to collide:

- ⇨ Velocity of the stars = **v**
- ⇨ Distance between the centre of the stars = **2R**

Total potential energy $=\frac{-GMM}{2R}$

Total energy of the two stars $=Mv^2-\frac{GMM}{2R}$

Using the law of conservation of energy: $Mv^2 - \frac{GMM}{2R} = \frac{-GMM}{r}$

$$\Rightarrow \quad v^2 = \frac{-GM}{r} + \frac{GM}{2R} = GM\left(-\frac{1}{r} + \frac{1}{2R}\right)$$

$$\Rightarrow \quad v^2 = 6.67\times10^{-11}\times2\times10^{30}\left[-\frac{1}{10^{12}} + \frac{1}{2\times10^{7}}\right]$$

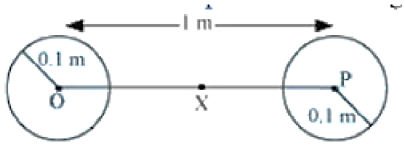
$$\Rightarrow \quad v^2 = 13.34\times10^{19}\left[-10^{-12} + 5\times10^{-8}\right]$$

$$\Rightarrow \quad v^2 \sim 6.67\times10^{12}$$

$$v = \sqrt{6.67\times10^{12}} = 2.58\times10^{6}\,\text{m/s}$$

8.21: Two heavy spheres each of mass 100 kg and radius 0.10 m are placed 1.0 m apart on a horizontal table. What is the gravitational force and potential at the mid point of the line joining the centres of the spheres? Is an object placed at that point in equilibrium? If so, is the equilibrium stable or unstable?

Solution: The situation is represented



Mass of each sphere: $M = 100\text{kg}$

Separation between the spheres: $r = 1\text{m}$

X is the location where the two spheres meet in the middle. At point X, the gravitational force will be zero. This is due to the fact that each sphere's gravitational force will act in different directions.

Gravitational potential at point X: $= \frac{-GM}{\left(\frac{r}{2}\right)} - \frac{GM}{\left(\frac{r}{2}\right)} = -4\frac{GM}{r}$

$$= \frac{4\times6.67\times10^{-11}\times100}{1}$$

$$= -2.67\times10^{-8}\,\text{J/kg}$$

Any object placed at point X will be in a state of balance, but the equilibrium will be unstable. This is due to the fact that any change in the object's position will change the effective force in that direction.

Additional Exercise

8.22: As you have learnt in the text, a geostationary satellite orbits the earth at a height of nearly $36,000$ km from the surface of the earth. What is the potential due to earth's gravity at

the site of this satellite? (Take the potential energy at infinity to be zero). Mass of the earth; $M_e = 6.0\times10^{24}\text{kg}$

Solution: Mass of the Earth**:** $M_e = 6.0\times10^{24}\text{kg}$

Radius of the Earth: $R = 6400\text{km} = 6.4\times10^6\text{m}$

Height of a geostationary satellite: $h = 36000\text{km} = 3.6\times10^7\text{m}$

Gravitational potential energy: $= \dfrac{-GM}{(R+h)} = -\dfrac{6.67\times10^{-11}\times6.0\times10^{24}}{3.6\times10^7+0.64\times10^7}$

$= -9.4\times10^6\text{J/kg}$

8.23: A star 2.5 times the mass of the sun and collapsed to a size of 12 km rotates with a speed of 1.2 rev. per second. (Extremely compact stars of this kind are known as neutron stars. Certain stellar objects called pulsars belong to this category). Will an object placed on its equator remain stuck to its surface due to gravity? (Mass of the sun $= 2\times10^{30}\text{kg}$).

Solution: If the inward gravitational pull is stronger than the outward centrifugal force induced by the star's spin, a body becomes stuck to its surface.

Gravitational force: $f_g = \dfrac{GMm}{R^2}$

where, $M =$ Mass of the star $= 2.5\times2\times10^{30} = 5\times10^{30}\text{kg}$

$R =$ Radius of the star $= 12\text{km} = 1.2\times10^4\text{m}$

$\therefore \quad f_s = \dfrac{6.67\times10^{-11}\times5\times10^{30}\times m}{\left(1.2\times10^4\right)^2} = 2.31\times10^{11}\text{mN}$

Centrifugal force: $f_c = mr\omega^2$

Angular speed: $= 2\pi v$

frequency $= 1.2$ rev s^{-1}

Now, $f_c = mR(2\pi v)^2 = m\times\left(1.2\times10^4\right)\times4\times(3.14)^2\times(1.2)^2 = 1.7\times10^5 m\text{N}$

The body will remain glued to the star's surface because $\text{fg} > \text{fc}$.

8.24: A spaceship is stationed on Mars. How much energy must be expended on the spaceship to launch it out of the solar system? Mass of the space ship $= 1000\text{kg}$;

mass of mars $= 6.4\times10^{23}\text{kg}$; radius of mars $= 3395\text{km}$; mass of the Sun $= 2\times10^{30}\text{kg}$; radius of the orbit of mars= $2.28\times10^8\text{kg}$

Solution: Mass of the spaceship: $m_s = 1000\text{kg}$

Mass of the Sun: $M = 2\times10^{30}\text{kg}$

Mass of Mars: $m_m = 6.4\times10^{23}\text{kg}$

Orbital radius of Mars: $R = 2.28\times10^{8}\text{kg} = 2.28\times10^{11}\text{m}$

Radius of Mars: $r = 3395\text{km} = 3.395\times10^{6}\text{m}$

$\text{G} = 6.67\times10^{-11}\text{m}^2\text{kg}^{-2}$ (Universal gravitational constant)

Now, Potential energy of the spaceship due to the gravitational attraction of the Sun: $= \frac{-\text{GMm}_s}{R}$

Because of Mars' gravitational attraction, the spaceship's potential energy is: $= \frac{-\text{G}M_m m_s}{r}$

The spaceship's velocity, and thus its kinetic energy, will be 0 because it is stationed on Mars.

Total energy: $= \frac{-\text{G}Mm_s}{R} - \frac{-\text{G}M_s m_m}{r} = -\text{G}m_s\left(\frac{M}{R}+\frac{m_m}{r}\right)$

The system is in a bound state if the sign is negative.

The amount of energy necessary to send the spaceship out of the solar system is equal to – (Total energy of the spaceship) $= \text{G}m_s\left(\frac{M}{R}+\frac{m_m}{r}\right)$

$$= 6.67\times10^{-11}\times10^{3}\times\left(\frac{2\times10^{30}}{2.28\times10^{11}}+\frac{6.4\times10^{23}}{3.395\times10^{6}}\right) = 6.67\times10^{-8}\times89.50\times10^{17}$$

$$= 596.97\times10^{9}$$

$$= 6\times10^{11}\text{J}$$

8.25: A rocket is fired 'vertically' from the surface of mars with a speed of 2 km s–1. If 20% of its initial energy is lost due to Martian atmospheric resistance, how far will the rocket go from the surface of mars before returning to it? Mass of mars $= 6.4\times1023\text{kg}$ **Radius of mars** $= 3395\text{km}; \text{G} = 6.67\times10^{-11}\text{Nm}^2\text{kg}^{-2}$

Solution: Initial velocity $v = 2\text{km/s} = 2\times10^{3}\text{m/s}$

- Mass of Mars: $M = 6.4\times10^{23}\text{kg}$
- Radius of Mars: $R = 3395\text{km} = 3.395\times10^{6}\text{m}$

$\text{G} = 6.67\times10^{-11}\text{m}^2\text{kg}^{-2}$ (Universal gravitational constant)

- Mass of the rocket $= m = h$
- Initial kinetic energy $= \frac{1}{2}mv^2$
- Initial potential energy $= \frac{-GMm}{R}$
- Total initial energy $= \frac{1}{2}mv^2 - \frac{GMm}{R}$

Only 80% of its kinetic energy aids in reaching a height if 20% of its initial kinetic energy is wasted due to Martian air resistance.

Total initial energy $= \frac{80}{100} \times \frac{1}{2}mv^2 - \frac{\mathrm{GMm}}{R} = 0.4mv^2 - \frac{\mathrm{GMm}}{R}$

Now, Maximum height $= h$

The rocket's velocity, and hence its kinetic energy, will be zero at this altitude.

Total energy of the rocket $= -\frac{GMm}{(R+h)}$

Applying the law of conservation of energy: $0.4mv^2 - \frac{GMm}{R} = \frac{-GMm}{(R+h)}$

$$\Rightarrow 0.4v^2 = \frac{\mathrm{GM}}{R} - \frac{GM}{R+h}$$

$$\Rightarrow 0.4v^2 = \mathrm{G}M\left(\frac{R+h-R}{R(R+h)}\right)$$

$$\Rightarrow 0.4v^2 = \frac{GMh}{R(R+h)}$$

$$\Rightarrow \frac{R+h}{h} = \frac{\mathrm{G}M}{0.4v^2R}$$

$$\Rightarrow \frac{R}{h} + 1 = \frac{\mathrm{G}M}{0.4v^2R}$$

$$\Rightarrow \frac{R}{h} = \frac{GM}{0.4v^2R} - 1$$

$$\Rightarrow h = \frac{R}{\frac{GM}{0.4v^2R} - 1} = \frac{0.4\mathrm{R}^2v^2}{\mathrm{G}M - 0.4v^2R}$$

$$= \frac{0.4 \times \left(3.395 \times 10^6\right)^2 \times \left(2 \times 10^3\right)^2}{6.67 \times 10^{-11} \times 6.4 \times 10^{23} - 0.4 \times \left(2 \times 10^3\right)^2 \times \left(3.395 \times 10^6\right)}$$

$$\Rightarrow h = \frac{18.442 \times 10^{18}}{42.688 \times 10^{12} - 5.432 \times 10^{12}} = \frac{18.442}{37.256} \times 10^6$$

$$h = 495 \times 10^3\,\mathrm{m} = 495\mathrm{km}$$