UP Secondary Examination, 2018

Mathematics

Time: $3\frac{3}{4}$ hours M.M: 70

GENERAL INSTRUCTIONS TO THE EXAMINEES:

1. There are in all seven questions on this question paper.
2. All the questions are compulsory.
3. In the beginning of each, the number of parts to be attempted has been clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start with the first question and proceed to the last one.
6. Do not waste your time over a question you cannot solve.

Q1. All questions are compulsory.

(a) What will be the greatest common divisor (GCD) of the expressions $(x-2)(x+4)^2$ and $(x-2)^2$?

(i) $(x+4)^2$

(ii) $(x-2)^2$

(iii) $(x-2)(x+4)^2$

(iv) $(x-2)$

Solution:

The largest positive integer which divides two or more integer without any remainder is called Highest Common Factor (HCF) or Greatest Common Divisor or Greatest Common Factor.

Given expression: $(x-2)(x+4)^2$ and $(x-2)^2$

The common factor between the two expressions is the smaller power of $x-2$, which is $x-2$.

Hence, the required HCF is $(x-2)$.

(b) Find the mode of 12, 20, 21, 29, 12, 29, 12, 29, 24 and 29.

Solution:

The mode is the number that appears most frequently.

Given data: 12, 20, 21, 29, 12, 29, 12, 29, 24 and 29

The number with the highest frequency is 29, which occurs 4 times.

Therefore, mode = 29

(c) Find the value of $\cos 15^{\circ} + \cos 105^{\circ}$.

(i) $\frac{1}{2}$

(ii) $\frac{1}{\sqrt{2}}$

(iii) $\frac{\sqrt{3}}{2}$

(iv) $\sqrt{\frac{3}{2}}$

Solution:

Given expression: $\cos 15^{0} + \cos 105^{0}$

The above expression can be written as $\cos\,(45^{\circ} - 30^{\circ}) + \cos\,(60^{\circ} + 45^{\circ})$.

We know that:

$$\cos(a-b) = \cos a \times \cos b + \sin a \times \sin b$$
$$\cos(a+b) = \cos a \times \cos b - \sin a \times \sin b$$
$$\cos(45^{\circ} - 30^{\circ}) + \cos\,(60^{\circ} + 45^{\circ})$$
$$= \cos 45^{\circ} \cos 30^{\circ} + \sin 45^{\circ} \sin 30^{\circ} + \cos 60^{\circ} \cos 45^{\circ} - \sin 60^{\circ} \sin 45^{\circ}$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}+1+1-\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$

Hence, the value is $\frac{1}{\sqrt{2}}$.

(d) Two circles touch each other internally. The distance between their centres is 3 cm. If the radius of one circle is 2 cm, what will be the radius of the other circle?

(i) 4 cm

(ii) 5 cm

(iii) 6 cm

(iv) 7 cm

Solution:

When two circles touch each other internally, the distance between their centres is equal to the difference between their radii.

Let the radius of the larger circle be RR and the radius of the smaller circle be r.

Given:

r = 2 cm

Distance between the centres = 3 cm

From the property of internally touching circles:

R – r = 3

Substitute r=2:

R – 2 = 3

Solve for R:

R = 3 + 2 = 5 cm

Therefore, the radius of the other circle is **5 cm**.

(e) What will be the slope of the line passing through the points (a, a^2) and (b, b^2)?

(i) 1

(ii) $\frac{b}{a}$

(iii) $(a + b)$

(iv) $-(a - b)$

Solution:

We have the point (a, a^2) and (b, b^2).

We know that:

$$M = \frac{Y_2 - Y_1}{X_2 - X_1}$$

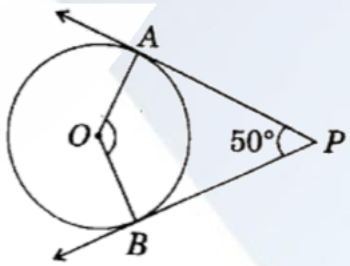
$$M = \frac{b^2 - a^2}{b - a}$$

$$M = \frac{(b-a)(b+a)}{(b-a)} = (b+a)$$

$$M = (a+b)$$

Hence, the slop is $(a+b)$.

(f) In the given figure, O is the centre of the circle. PA and PB are tangents to the circle, touching it at points A and B, respectively. If ∠APB = 50°, find the measure of ∠AOB.



(i) 100°

(ii)105°

(iii)120°

(iv)130°

Solution:

We know that:

In a circle, the angle subtended by two tangents at the external point is supplementary to the angle subtended by the same points at the centre.

$\angle APB + \angle AOB = 180^o$

Given: $\angle APB = 50^o$

Therefore, $\angle AOB = 180^o - 50^o = 130^o$

Q2. All questions are compulsory.

(a). The slope of a line is $-\frac{b}{a}$, and it intersects the positive direction of the y −axis at b. Find the equation of the line.

Solution:

The slope of the line is $m = -\frac{b}{a}$, and the y-intercept is b. Using the slope-intercept form of a line, $y = mx + c$, where c is the y-intercept:

$$y = -\frac{b}{a}x + b$$

Rewriting it in standard form:

$$bx + ay = ab$$

Hence, the equation of the line is: $bx + ay = ab$

(b) Find the value of $\tan\frac{13\pi}{6}$.

Solution:

The given expression is $\tan\frac{13\pi}{6}$.

Since the tangent function has a period of 2π, subtract 2π from $\frac{13\pi}{6}$:

$$\tan\left(\frac{13\pi}{6} - 2\pi\right)$$

$$\tan\frac{13\pi - 12\pi}{6} = \tan\frac{\pi}{6}$$

$$\tan\ 30^\circ = \frac{1}{\sqrt{3}}$$

Thus, the value of $\tan\frac{13\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

(c). The median of the numbers arranged in descending order: 48, 44, 41, 36, (2x+8), (2x-6), 14, 11, 8, 6 is 25. Find the value of x.

Solution:

The given numbers are arranged in descending order as:

48, 44, 41, 36, (2x+8), (2x-6), 14, 11, 8, 6.

The median of these 10 numbers will be the average of the 5th and 6th numbers.

So, the 5th and 6th numbers are 2x+8 and 2x-6, respectively.

The median is given as 25:

$$25 = \frac{(2x+8)+(2x-6)}{2}$$

$$50 = 4x + 2$$

$$4x = 48$$

$$x = 12$$

(d). If $f(x) = \frac{x^2-4x+13}{x+1}$, find the value of $f(8)$.

Solution:

Simplify the expression for $f(8)$.

$$f(8) = \frac{(8)^2 - 4(8) + 13}{8+1}$$

$$f(8) = \frac{64 - 32 + 13}{9} = \frac{45}{9}$$

$$f(8) = 5$$

Hence, the value of $f(8) = 5$.

Q3. All questions are compulsory.

(a) Prove that:

$$\frac{\cos(60^\circ - A) + \sin(30^\circ - A)}{\cos(30^\circ - A) - \sin(60^\circ - A)} = \cot A$$

Solution:

L.H.S. = $\frac{\cos(60^\circ - A) + \sin(30^\circ - A)}{\cos(30^\circ - A) - \sin(60^\circ - A)}$

$$= \frac{\cos(90^\circ - (30^\circ + A)) + \sin(30^\circ - A)}{\cos(90^\circ - (60^\circ + A)) - \sin(60^\circ - A)}$$

$$= \frac{\sin(30^\circ + A) + \sin(30^\circ - A)}{\sin(60^\circ + A) - \sin(60^\circ - A)}$$

We know that,

$$\text{Sin}C + \text{Sin}D = 2\text{Sin}\frac{(C+D)}{2}\text{Cos}\frac{C-D}{2}$$

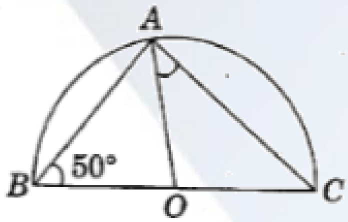
$$\text{Sin}C - \text{Sin}D = 2\text{Sin}\frac{(C+D)}{2}\text{Cos}\frac{C-D}{2}$$

$$=\frac{2\sin\frac{(30^{\circ}+A+30^{\circ}-A)}{2}\cos\frac{(30^{\circ}+A-30^{\circ}+A)}{2}}{2\cos\frac{(30^{0}+A+30^{\circ}-A)}{2}\sin\frac{(30^{\circ}+A-30^{\circ}+A)}{2}}$$

$$=\frac{2\sin\ 30^{\circ}\cos\ A}{2\sin\ 30^{\circ}\sin\ A}=\cot\ A$$

(b). figure in, find the angle $\angle OAC$.whose hemisphere of center is O and diameter is BC. If the angle $\angle ABO=50^{\circ}$.

In the diagram, the centre of the semicircle is O, and the diameter is BC. If ∠ABO = 50°, then find the measure of ∠OAC.



Solution:

We know that the diameter of a circle makes a 90° angle at any point on the circle's circumference.

Therefore, $\angle\mathrm{BAC}=90^{\circ}$

In triangle ABC,

$\angle\mathrm{ABC}+\angle\mathrm{BAC}+\angle\mathrm{ACB}=180^{\circ}$

$\angle 50^{\circ}+90^{\circ}+\angle\mathrm{ACB}=180^{\circ}$

$\angle\mathrm{ACB}=180^{\circ}-140^{\circ}$

$\angle\mathrm{ACB}=40^{\circ}$

$\angle\mathrm{ACO}=40^{\circ}$

Since OC = OA [Radius]

Therefore, $\angle\mathrm{ACO}=\angle\mathrm{OAC}=40^{\circ}$

(c). Find the positive slope of the line $x+y=9$ with respect to the x-axis and the segment intercepted by the line on the y-axis.

Solution:

The given line is $x + y = 9$.

Rewrite the equation in slope-intercept form: $y = -x + 9$.

The slope of the line is $m = -1$.

Set $x = 0$ in the equation: $0 + y = 9$. So, the y-intercept is $y = 9$.

Thus, the positive slope of the line is -1 and the segment intercepted by the line on the y-axis is 9 units.

(d) Find the height of a cylinder, whose curved surface area is 900π cm^2 and its radius 10 cm.

Solution:

We know that:

The curved surface area of cylinder is $2\pi rh$.

$2\pi rh = 900\pi$

$2\pi \times 10 \times h = 900\pi$

$2 \times 10 \times h = 900$

$h = 45$ cm

Hence, the value of $h = 45$ cm.

Q4. All questions are compulsory.

(a) Sanjay buys a bicycle for ₹3,528, including sales tax. If the marked price of the bicycle is ₹3,360, find the rate of sales tax.

Solution:

Marked Price (MP) = ₹3,360

Selling Price (SP) = ₹3,528

Sales Tax = SP - MP = ₹3,528 - ₹3,360 = ₹168

Sales Tax Rate $= \frac{Sales\ Tax}{Marked\ Price} \times 100$

$= \frac{168}{3360} \times 100$

$= 5\%$

Therefore, Sales tax rate is 5%.

(b). Prove that: $\cot A - \tan A = 2\cot 2A$.

Solution:

We have the expression $\cot A - \tan A = 2\cot 2A$.

We take L.H.S.

$$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\sin A \cdot \cos A}$$

We know that:

$\cos^2 A - \sin^2 A = \cos 2A$

Simplify further,

$$= \frac{\cos 2A}{\sin A \cos A}$$

$$= \frac{2\cos 2A}{2\sin A \cos A} \quad (2\sin a \cos a = \sin 2a)$$

$$= \frac{2\cos 2A}{\sin 2A}$$

$= 2cot2A$

$= R.H.S.$

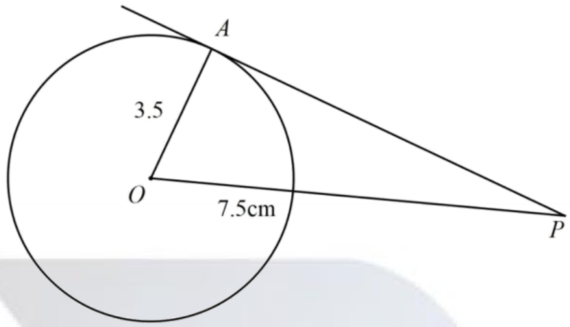
Hence proved.

(c). Draw a circle with a radius of 3.5 cm. Construct tangents to the circle from a point located 7.5 cm away from the centre of the circle.

Solution:

We have a radius OA is 3.5 and OP is 7.5 cm.

Draw in figure:



(d). The height and radius of a right circular cone are 15 cm and 8 cm, respectively. Find the curved surface area of the cone. (Take π = 3.14).

Solution:

We know that the formula of curved surface area of a cone is $\pi r l$.

$l = \sqrt{15^2 + 8^2}$

$l = \sqrt{225 + 64} = \sqrt{289}$

$l = 17$ cm

$\pi r l = 3.14 \times 8 \times 17 = 427.04$ cm

Hence, the curved surface area of the cone is 427.04 cm.

Q5. All questions are compulsory.

(a) Find the L.C.M. (Lowest Common Multiple) of the expressions $x^2 + x - 12, x^2 - 9$, and $x^2 + 8x + 16$.

Solution:

Factorize the expressions:

$x^2 + x - 12$

$= x^2 + 4x - 3x - 12$

$= x(x + 4) - 3(x + 4)$

$= (x + 4)(x - 3)$

$x^2 - 9 = (x + 3)(x - 3)$

$x^2 + 8x + 16$

$= x^2 + 4x + 4x + 16$

$= x(x + 4) + 4(x + 4)$

$= (x + 4)(x + 4)$

$= (x + 4)^2$

Take each unique factor with the highest power:

Therefore, L.C.M= $(x + 4)^2(x - 3)(x + 3)$

(b). Prove that:

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = 1 + \sec A \operatorname{cosec} A$$

Solution:

First, take L.H.S.

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$

$$= \frac{\tan A}{1 - \frac{1}{\tan A}} + \frac{\cot A}{1 - \tan A}$$

$$= \frac{\tan^2 A}{1 - \tan A} - \frac{\cot A}{\tan A - 1}$$

$$= \frac{\tan^2 - \cot A}{\tan A - 1}$$

$$= \frac{\frac{\sin^2}{\cos^2} - \frac{\cos A}{\sin A}}{\frac{\sin A}{\cos A} - 1}$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin A \cos A(\sin A - \cos A)}$$

$$= \frac{(\sin^2 + \cos^2)(\sin A - \cos A) + \cos A \sin 2A - \sin A \cos A}{\sin A \cos A(\sin A - \cos A)}$$

$$= \frac{(\sin^2 + \cos^2)(\sin A - \cos A) + \sin A \cos A(\sin A - \cos A)}{\sin A \cos A(\sin A - \cos A)}$$

$$= \frac{(\sin^2 + \cos^2) + \sin A \cos A}{\sin A \cos A}$$

$$= \frac{(\sin^2 + \cos^2)}{\sin A \cos A} + 1$$

$$= 1 + \tan A + \cot A$$

$$= 1 + \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= 1 + \frac{1}{\sin A \cos A}$$

$$= 1 + \operatorname{cosec} A \cdot \sec A = R.H.S.$$

Hence, proved.

(c). Find the equation of a line parallel to $3x - 5y = 4$ that passes through the midpoint P of the line segment joining the points (6, -4) and (4, 4).

Solution:

Find the midpoint P:

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6 + 4}{2}, \frac{-4 + 4}{2}\right) = (5,0)$$

Equation of line parallel to $\mathbf{3x - 5y = 4}$:

Parallel lines have the same slope, so the equation is of the form $3x - 5y + c = 0$.

Substitute P (5,0) into the equation:

$3(5) - 5(0) + c = 0$

$\Rightarrow 15 + c = 0$

$\Rightarrow c = -15$

Final equation:

The required equation is $3x - 5y - 15 = 0$.

d). In the financial year 2016-17, an individual's annual income (excluding house rent allowance) is ₹6,20,000. He deposits ₹8,000 per month into his General Provident Fund (GPF) account and ₹80,000 into his Public Provident Fund (PPF) account. Calculate the income tax payable by the individual, given that the maximum deduction allowed on savings is ₹1,50,000.

The applicable income tax rates are as follows:

Income	Income tax
Up to Rs. 2,50,000	zero
Rs 2,50,001 to 5,00,000	Rs 2,50,000 increase income of 10%

Education extra income tax 3%.

Solution:

Total income = 6, 20,000

Saving income $8000 \times 12 = $ Rs 96,000

P. F = Rs. 80,000

Discount $= 96{,}000 + 80{,}000 + 150{,}000 =$ Rs 3,26,000

$620{,}000 - 326{,}000 = 294{,}000$

Rate of income tax.

$= 294{,}000 - 250{,}000 = 44{,}000$
$= 44{,}000 \times 10\%$
$= 44{,}000 \times \frac{10}{100} = 4400$
$= 4400 \times 3\%$
$= 4400 \times \frac{3}{100} = 132$

Pay education of income tax $= 4400 + 132 =$ Rs. 4532

Hence, the value of income tax of education is Rs 4532.

Q6. All questions are compulsory.

(a). Find the median from the following frequency distribution.

Class interval	$0-10$	$10-20$	$20-30$	$30-40$	$40-50$
Frequency s	12	18	24	15	11

Solution:

We know that: $x_1 = \frac{0+10}{2} = 5$; Similarly, $x_2, x_{3---}x_5$

C.I	f_i	x_i	$f_i x_i$
$0-10$	12	5	60
$10-20$	18	15	270
$20-30$	24	25	600
$30-40$	15	35	575
$40-50$	11	45	495
	$\sum f_i = 80$		$\sum f_i x_i = 2{,}000$

Mean $= \frac{2,000}{80} = 25$

Hence, the value of the mean is 25.

(b). A toy is in the shape of a cone mounted on a hemisphere with a radius of 3.5 cm. The total height of the toy is 15.5 cm. Find the total surface area of the toy.

Solution:

Radius of hemisphere = 3.5 cm

Total height of toy = 15.5 cm

Height of cone = 15.5 − 3.5 = 12 cm

We get the slant height of cone,

$$l = \sqrt{r^2 + h^2}$$
$$= \sqrt{\left(\frac{7}{2}\right)^2 + 12^2} = \frac{25}{2} \text{ cm}$$

The total surface area of the toy = Curved surface of the hemisphere + Curved surface of the cone

$$(2\pi r^2 + \pi r l) \text{ cm}^2$$
$$= (\pi \times r \times (2r + l)) \text{ cm}^2$$
$$= \left\{\frac{22}{7} \times \frac{7}{2} \times \left(7 + \frac{25}{2}\right)\right\} \text{cm}^2$$
$$= \frac{429}{2} \text{ cm}^2 = 214.5 \text{ cm}^2$$

Hence, the total surface area of the toy is 214.5 cm^2.

(c). From a point on the ground, the angles of elevation to the base and the top of a communication tower placed on the roof of a 20-meter-high building are 45° and 60°, respectively. Find the height of the tower. (Take $\sqrt{3}$ =1.732).

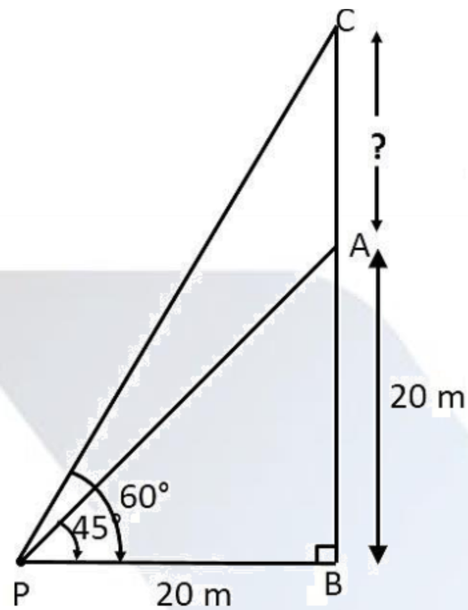
Solution:

In $\triangle PBA$,

$$\tan 45^\circ = \frac{AB}{PB} = \frac{20}{x}$$

$$1 = \frac{20}{x}$$

$$x = 20 \text{ cm}$$



In $\triangle PBC$,

$$\tan\ 60^\circ = \frac{BC}{PB}$$

$$\tan\ 60^\circ = \frac{20 + x}{20}$$

$$\sqrt{3} = \frac{20 + x}{20}$$

$$20 + x = 20\sqrt{3}$$

$$x = 20\sqrt{3} - 20$$

$$x = 20(\sqrt{3} - 1)$$

Thus, the height of the tower is $20(\sqrt{3} - 1)$ m.

Q7. All questions are compulsory.

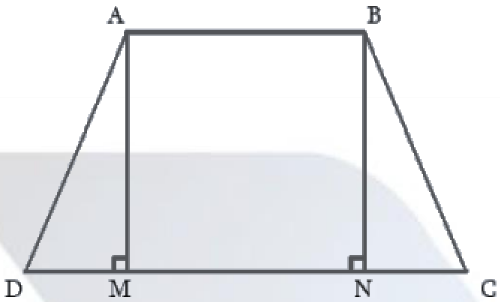
(a) Prove that if the non-parallel sides of a trapezium are equal, then it will be a cyclic quadrilateral.

Solution:

We know that, if the sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.

Draw a trapezium ABCD with AB || CD.

AD and BC are the non-parallel sides that are equal. AD = BC. Draw AM ⊥ CD and BN ⊥ CD.



Consider ΔAMD and ΔBNC

AD = BC (Given)

∠AMD = ∠BNC (90°)

AM = BN (Perpendicular distance between two parallel lines is same)

By RHS congruence, ΔAMD ≅ ΔBNC.

Using CPCT, ∠ADC = ∠BCD.....(1)

∠BAD and ∠ADC are on the same side of transversal AD.

∠BAD + ∠ADC = 180°

∠BAD + ∠BCD = 180° [From equation (1)]

This equation proves that the sum of opposite angles is supplementary.

Hence, ABCD is a cyclic quadrilateral.

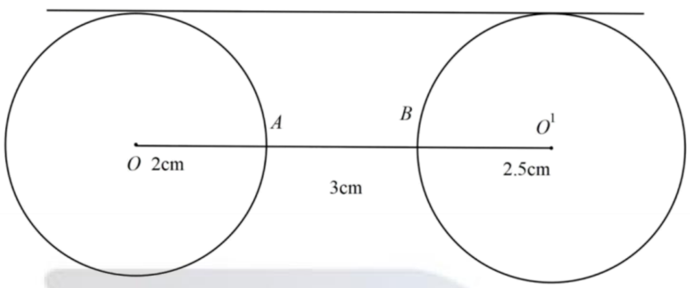
Or

Draw two circles with radii of 2 cm and 2.5 cm, respectively, with a distance of 7.5 cm between their centers. Construct the common external tangents of these circles and measure their length. Confirm your answer through calculation.

Solution:

We have two circles with radii of 2 cm and 2.5 cm, and the distance between their centers is 7.5 cm.

We have $OA = 2$ cm and $O'B = 2.5$ cm and $OO' = 7.5$ cm.



(b) The diagonal of the rectangular field is 30 m more than the shortest side of the field. If the larger side is 15 m more than the shortest side. Then find the sides of the rectangular field.

Solution:

Let the small side of x .

we get larger side is x + 15.

And diagonal is $x + 30$.

We know that the rectangular diagonal is:

$D = \sqrt{l^2 + b^2}$

$30 + x = \sqrt{(15 + x)^2 + x^2}$

$(30 + x)^2 = (15 + x)^2 + x^2$

$900 + 60x + x^2 = 225 + x^2 + x^2$

$x^2 + 30 - 60x + 225 - 900$

$x^2 - 30x - 625 = 0$

$x^2 + 15x - 45x - 675 = 0$

$x(x - 45) - 45(x + 15)$

$= (x - 45)(x + 15)$

$x = 45$

$x = -15$

We get the value of x is $x = 45$.

Since, the small side is $x = 45$ and largest side is $x = 60$.

Or

(i) Find the roots of quadratic equation $3x^2 - 6x + 2 = 0$.

Solution:

$3x^2 - 6x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6}$$

$$x = \frac{6 \pm 2\sqrt{3}}{6}$$

Hence, the root of quadratic equation is $x = \frac{6 \pm 2\sqrt{3}}{6}$.

(ii) If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β, find the value of the expression $\alpha^3 + \beta^3$.

Solution:

We know that:

The sum of root $\alpha + \beta = -\frac{b}{a}$.

The product of root $\alpha \cdot \beta = \frac{c}{a}$.

We know that:

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= (\alpha + \beta)[(\alpha + \beta^2) - \alpha\beta - 2\alpha\beta]$

$$= \left(\frac{-b}{a}\right)\left[\left(\frac{-b}{b}\right)^2 - 3\alpha\beta\right]$$

$$= \left(\frac{-b}{a}\right)\left[\left(\frac{-b}{b}\right)^2 - 3\left(\frac{c}{a}\right)\right]$$

$$= \left(\frac{-b}{a}\right)\left[\frac{b^2}{a^2} - \frac{3c}{a}\right]$$

$$= \left(\frac{-b}{a}\right)\left[\frac{b^2 - 3ca}{a^2}\right]$$

$$= \frac{-b^2 + 3abc}{a^3}$$

Hence, the value of $\alpha^3 + \beta^3$ is $\frac{-b^2+3abc}{a^3}$.